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1990 J. Phys. A: Math. Gen. 23 2327

(http://iopscience.iop.org/0305-4470/23/12/017)

J. Phys. A: Math. Gen. 23 (1990) 2327–2362. Printed in the UK

A direct approach to studying soliton perturbations

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Received 24 January 1990

Abstract. Starting with an integrable nonlinear evolution equation, we investigate perturbations about a one-soliton solution, through the inversion of a linear equation for the first-order correction to the soliton solution. This inversion differs from past methods, as the proposed method takes place in coordinate space, not spectral space, while it employs some of the tools of inverse scattering theory. The method is applied to the Korteweg–deVries, nonlinear Schrödinger and sine-Gordon equations. The first-order corrections are then obtained.

1. Introduction

In this paper we investigate the use of a direct approach to studying perturbed nonlinear evolution equations. The general procedure consists of expanding about a solution of the given unperturbed evolution equation, leading to a linear partial differential equation to be solved for the first-order correction. This equation can be solved through the use of an expansion of the solution in an appropriate set of basis states. The states corresponding to discrete eigenvalues turn out to lead to secular behaviour, and they can be controlled by placing conditions on the soliton parameters. This method will be demonstrated for perturbations of the Korteweg–deVries equation (KdV), the nonlinear Schrödinger equation (NLS), and the sine-Gordon equation in lightcone coordinates (SG).

We are interested in the effects of weak perturbations of the equations, which are exactly solvable by the inverse scattering transform (IST). If we measure the strength of the perturbation by ϵ, then by weak we mean that $0 < \epsilon \ll 1$. Such perturbations can be studied directly in coordinate space, or the effects of the perturbations on the scattering data can be studied in spectral space.

We refer to the first as the *direct* method. For example, one can use the method of *multiple scales*, or the *derivative expansion* method [20]. In this case the independent variable t is transformed into several variables by

$$t_n = \epsilon^n t \qquad n = 0, 1, 2, \ldots \tag{1}$$

where each t_n is an order of ϵ smaller than the previous time. The time derivatives are replaced by the expansion

$$\partial_t = \sum_{n=0}^{\infty} \epsilon^n \, \partial_{t_n} \tag{2}$$

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0305-4470/90/122327+36$03.50

thus, the name *derivative expansion*. At the same time the dependent variable is expanded in an asymptotic series

$$u = \sum_{n=0}^{\infty} \epsilon^n u_n. \tag{3}$$

These expressions are inserted into the equation of interest. Equating the coefficients of each order in ϵ, one obtains a sequence of equations to be solved for u_n.

As each order is solved, one has to ensure that there are no secularities in the solution, i.e. that the solution does not blow up in time, or become non-uniform in x. To guarantee this, conditions are placed on any free parameters which are at hand, often leading to the dependence of these parameters on the slow time scale. In the following we will expand certain soliton parameters, such as the amplitude and soliton centre. For example, for the amplitude we could write

$$A = A_0 + \epsilon A_1 + \dots \tag{4}$$

and then assume that each term in this expansion depends on a slow time scale. The dependence on this time can then be determined in such a way as to eliminate any potential secularities in the solution. The equations used to determine this behaviour will be referred to as *secularity conditions.*

This approach differs from the methods based on the perturbation of certain *scattering data* [22–26, 30, 40, 41], as the proposed method takes place in coordinate space and not spectral space. However, it also differs from many of the usual direct methods in that some of the tools of inverse scattering theory for the underlying nonlinear equation are used. Using this method, one can easily see how the secularities develop, obtain the first-order correction to the solution of the perturbed equation, and even study perturbations involving initial conditions that are close to a single-soliton profile.

We should note that Kaup [28, 29] has used a similar approach in studying a perturbed sine-Gordon equation. In that work there was no need to rely on the tools of inverse scattering, due to the approximations made. Several other authors have also studied soliton perturbations, using a direct approach without employing IST [4, 34, 35]. However, in all of these cases the authors use a *quasi-stationary* assumption, leading to results which are valid for short times, or small distances. Keener and McLaughlin [31, 32] have used a direct approach by obtaining the appropriate Green functions for the nonlinear Schrödinger and sine-Gordon equations. In their work they do depend on inverse scattering. At least in these two problems, our results agree. However, there has not been, to our knowledge, a satisfactory study of a direct perturbation of the KdV equation.

In section 2 we present the steps involved in the proposed method. In the following sections we apply the method to three common integrable evolution equations. The method is tested on a damped KdV equation in some detail, showing how second-order terms can result in a first-order change in the soliton position. As other researchers have seen, the soliton decays exponentially, its width broadens, and the soliton is seen to slow down. At the same time a tail is formed, consisting of a shelf and decaying oscillations.

2. The general method

Ichikawa, Iino and Wadati [17–19] have shown that Lax-pair operators for the squared eigenfunctions of the AKNS scheme [2] can be constructed using the Chen–Lee–Liu algorithm [6] for the KdV, mKdV, SG and NLS equations. Examining the time evolution operator and its adjoint, we are led to the linearised version of the particular equation of interest. Namely, we find that a particular combination of squared eigenfunctions is the solution of the homogeneous equation. Using the method of variation of parameters, we can assume that the first-order solution of a perturbation problem is a linear combination of the independent solutions of the (linearised) homogeneous equation. The expansion coefficients are then obtained by taking inner products with the adjoint states. These adjoint states are found through the use of the spectral operator, and its adjoint, for the squared eigenstates, which can also be used to obtain the required orthogonality relations between the two sets of states.

It should be noted that the spectral operator above is more commonly referred to as the recursion operator [9]. This operator and the operator from the linearised evolution equation form another Lax pair for the nonlinear evolution equation [6, 45]. Knowing the eigenfunctions of the recursion operator and their connection with the original spectral problem, one can construct the appropriate basis for the perturbation theory.

Before continuing to the details we outline the procedure, which will be used.

I. Perturbation expansion

A We first obtain the linearised equation for the nonlinear equation under study by using an expansion $u = u_0 + \epsilon u_1 + \ldots$. One may also need derivative expansions for the time, or other variables, which will be useful in eliminating any secular growth that may result.

II. Perturbation basis

A By using the associated Lax pair for a given evolution equation, the time evolution of the squared eigenfunctions may be determined.

B Now we use these results to get the correct combination of squared eigenfunctions, which satisfies the homogeneous linearised equation.

C Once the correct form is known for the squared eigenfunctions, the adjoint states should be found. Firstly, find the appropriate operator to set up an eigenvalue problem, and then search for the solutions to the adjoint problem. These will be useful for determining completeness properties and orthogonality relations. This can easily be seen from an example. From II.A we have the solution, Ω, of the homogeneous form of the linearised equation, which satisfies the eigenvalue problem

$$L\Omega = \lambda\Omega. \tag{5}$$

We find the adjoint state, Ω^{A} and its associated eigenvalue problem:

$$L^{\mathrm{A}}\Omega^{\mathrm{A}} = \lambda'\Omega^{\mathrm{A}}. \tag{6}$$

Multiplying the first equation by Ω^{A} and the second by Ω, and then subtracting, we have

$$(\lambda - \lambda')\Omega\Omega^{\mathrm{A}} = \Omega^{\mathrm{A}}L\Omega - \Omega L^{\mathrm{A}}\Omega^{\mathrm{A}}. \tag{7}$$

Hopefully, one can rewrite the right-hand side as a divergence, in the same manner as Kaup has done for the states associated with the Zakharov–Shabat problem [27]. Integrating over x, one obtains

$$(\lambda - \lambda')\langle \Omega | \Omega^{A} \rangle = f|_{-\infty}^{\infty} \qquad f_x = \Omega^{A} L \Omega - \Omega L^{A} \Omega^{A}. \tag{8}$$

Using the asymptotic behaviour of the Jost solutions, the orthogonality relation between Ω and Ω^{A} can be obtained.

III. Inversion of linear operator

A Now, insert an expansion in the independent solutions of the homogeneous linearised equation into the first-order equation, using the known completeness of the states. Here we need to have all of the independent solutions of the homogeneous problem, i.e. the homogeneous linearised equation. We have the independence of the states from the orthogonality relations, but to ensure that we have all of them, we need a completeness statement. This is easy for the cases to be considered, as we only need to show how our expansion is related the known results [27, 40, 44]. However, for other problems this would have to be proved.

B Use the orthogonality relations to solve for the expansion coefficients, thus obtaining the desired answer. Here we assume that u_1 can be expanded in the above complete set, and then we solve for the expansion coefficients. As an example, we treat a special case. Let the first-order equation be given by

$$\mathcal{L} u_1 = F_1. \tag{9}$$

Assuming u_1 to be of the form

$$u_1 = \int f \Omega \, d\lambda \tag{10}$$

then operating on it by $\mathcal{L}$, might give

$$F_1 = \int f_t \Omega \, d\lambda. \tag{11}$$

Now, multiplying by the adjoint and integrating over x will lead to equations for $f(\lambda)$. These equations can be solved for the expansion coefficients, using the appropriate initial conditions.

IV. Secularity conditions

Finally, we insert these results in the expansion for u_1 and demand that u_1 be bounded in t. This can lead to certain *secularity* conditions, which determine how the soliton shape and position are affected by the perturbation. In general, these are related to the bound states of the linear operator $\mathcal{L}$.

3. The perturbed KdV equation

We first consider the direct approach to solving the perturbed KdV equation

$$u_t + 6uu_x + u_{xxx} = \epsilon F[u] \tag{12}$$

subject to the initial condition

$$u(x,0) = 2\eta^2 \operatorname{sech}^2 \eta x. \tag{13}$$

For small perturbations, we expect that the solution will remain close to the soliton solution for some time. Therefore, the solution we seek will be roughly a soliton with a slowly changing shape and location plus a correction.

3.1. Perturbation expansion

We assume an asymptotic expansion of the form

$$u(x,t) = u_0(x,t) + \epsilon u_1(x,t) + \dots \tag{14}$$

where we take

$$u_0(x,t) = 2\eta^2 \operatorname{sech}^2 \eta \left(x - \frac{1}{\epsilon} x_0 - x_1 \right). \tag{15}$$

Defining the two time scales, $T = t$ and $\tau = \epsilon t$, we assume that the soliton parameters η, x_0 and x_1 depend only on the slow scale τ.

Introducing the expansion (14) and the two time scales into equation (12), we obtain an expansion of (12) in powers of ϵ. Setting the coefficients of each order of ϵ to zero, we obtain a system of equations to be solved for u_n. The lowest-order equation is found as

$$24\eta_0^5 v v_\phi - 2\eta_0^3 x_{0\tau} v_\phi + 2\eta_0^5 v_{\phi\phi\phi} = 0 \tag{16}$$

where we have defined

$$v = \operatorname{sech} \phi \qquad \phi = \eta \left(x - \frac{1}{\epsilon} x_0 - x_1 \right). \tag{17}$$

This equation will be satisfied if and only if

$$x_{0\tau} = 4\eta^2. \tag{18}$$

The first-order equation then becomes

$$\mathcal{L} u_1 = -4\eta\eta_\tau v - 2\eta\eta_\tau \phi v_\phi + 2\eta^3 x_{1\tau} v_\phi + F[u_0] \tag{19}$$

where $\mathcal{L}$ is the linearised KdV operator

$$\mathcal{L} \equiv \partial_T - 4\eta^3 \partial_\phi + 6\eta \partial_\phi u_0 + \eta^3 \partial_\phi^3. \tag{20}$$

3.2. Perturbation basis

The problem is now to invert this operator. We discuss the details of this inversion for the general problem

$$\mathcal{L} u_1 = \mathcal{F} \tag{21}$$

where $\mathcal{L}$ is given in (x,t) coordinates by

$$\mathcal{L} \equiv \partial_t + 6\partial_x u_0 + \partial_x^3. \tag{22}$$

This can be done by noting the relationship between the KdV equation, the associated 'Lax pair', and the linearised operator we are seeking to invert. Having established this relation, we will then proceed to discuss the inversion of the operator for the special case of $u_0 = 2\eta^2 \operatorname{sech}^2 \eta(x - \bar{x})$.

It is well known that the KdV equation ($q = u_0$)

$$q_t + 6qq_x + q_{xxx} = 0 \tag{23}$$

is an integrability condition for the equations [3]

$$v_{xx} + (\lambda^2 + q)v = 0 \tag{24}$$

$$v_t + v_{xxx} + 3(q - \lambda^2)v_x = \gamma v. \tag{25}$$

Namely, $v_{txx} = v_{xxt}$ provided q satisfies (23) and $\lambda_t = 0$. The constant γ is determined from the assumed asymptotic behaviour of v in the regions where q vanishes. In particular, if we assume that $v \sim e^{i\lambda x}$ ($v \sim e^{-i\lambda x}$) as $x \to \infty(-\infty)$, then $\gamma = -4i\lambda^3$ ($4i\lambda^3$). Keeping with standard notation [3], we will denote such Jost solutions as ψ_2 (ϕ_2).

If we operate on the function $f = \partial_x(v^2)$ with $\mathcal{L}$ from (22), using equations (24) and (25), we find that f satisfies the eigenvalue problem

$$\mathcal{L}f = 2\gamma f. \tag{26}$$

These eigenfunctions can be used as a basis in which to expand the solution of equation (21). The unknown expansion coefficients can then be determined using orthogonality relations between the basis functions and their adjoints.

In the Schrödinger eigenvalue problem there is a continuous spectrum for $\lambda^2 > 0$ and possible bound states for $\lambda^2 < 0$. The eigenstates for the continuous spectrum of $\mathcal{L}$ are easily found from these λ; however, the bound states $\partial_x v^2|_{\lambda_k}$ are not sufficient to complete the set of states [26, 27, 40]. We find the states we require from the work of Sachs [44]. If $f(x)$ is continuous and L^1, and if q satisfies

$$\| q \|_{L^1_2} \equiv \int_{-\infty}^{\infty} (1 + x^2) q(x)\, dx < \infty \tag{27}$$

then Sachs shows that $f(x)$ can be expanded as

$$f(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi i \lambda a^2(\lambda)} \Phi^A(x,\lambda)\Phi(y,\lambda) f(y)\, dy\, d\lambda + \frac{ic_1}{2\alpha_1}\int_{-\infty}^{\infty} [\Phi_1^A(x)\Lambda_1(y) - \Lambda_1^A(x)\Phi_1(y)] f(y)\, dy \tag{28}$$

where

$$\alpha_1 \equiv \frac{\psi_2(x,\lambda_1)}{\phi_2(x,\lambda_1)} \qquad c_1 \equiv \left(\frac{i}{2\alpha_1}\int_{-\infty}^{\infty} \Phi_1^A(x)\Lambda_1(x)\, dx\right)^{-1} \tag{29}$$

$$a(\lambda) = \frac{\lambda - i\eta}{\lambda + i\eta} \qquad \lambda_1 = i\eta. \tag{30}$$

Here we have borrowed some notation from Newell [40].

The complete set for perturbations about a one-soliton solution is given by $\{\Phi^A, \Phi_j^A, \Lambda_j^A\}$. These basis states are related to the Schrödinger eigenfunctions by

$$\Phi^A(x,t;\lambda) \equiv \partial_x \psi_2^2 \qquad \Phi_1^A(x,t) \equiv \partial_x \psi_2^2|_{\lambda_1} \tag{31}$$

$$\Lambda_1^A(x,t) \equiv \partial_\lambda \partial_x \psi_2^2|_{\lambda_1} - \frac{1}{\alpha_1^2}\partial_\lambda \partial_x \phi_2^2|_{\lambda_1}. \tag{32}$$

The *adjoint* states to these are given by

$$\Phi(x,t;\lambda) \equiv \phi_2^2 \qquad \Phi_1(x,t) \equiv \phi_2^2|_{\lambda_1} \tag{33}$$

$$\Lambda_1(x,t) \equiv \partial_\lambda \phi_2^2|_{\lambda_1} - \alpha_1^2 \partial_\lambda \phi_2^2|_{\lambda_1}. \tag{34}$$

Before returning to the inversion of the linearised KdV equation we need two properties of the basis states. Firstly, equation (26) gives the result of operating on the basis states with $\mathcal{L}$:

$$\begin{aligned} \mathcal{L}\Phi^{\mathrm{A}} &= -8\mathrm{i}\lambda^3\Phi^{\mathrm{A}} \\ \mathcal{L}\Phi_1^{\mathrm{A}} &= -8\mathrm{i}\lambda_1^3\Phi_1^{\mathrm{A}} \\ \mathcal{L}\Lambda_1^{\mathrm{A}} &= -8\mathrm{i}\lambda_1^3\Lambda_1^{\mathrm{A}} - 48\mathrm{i}\lambda_1^2\Phi_1^{\mathrm{A}}. \end{aligned} \tag{35}$$

Secondly, the non-zero inner products between the basis states and the adjoint states are found as:

$$\langle\Phi^{\mathrm{A}}(\lambda)|\Phi(\lambda')\rangle = 2\pi\mathrm{i}\lambda a^2(\lambda)\delta(\lambda-\lambda') \tag{36}$$

$$\langle\Lambda_1^{\mathrm{A}}|\Phi_1\rangle = \langle\Phi_1^{\mathrm{A}}|\Lambda_1\rangle = \mathrm{i}/2\eta \tag{37}$$

where we define the inner product by

$$\langle f(x)|g(x)\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{d}x. \tag{38}$$

3.3. Inversion of linear operator

We can now proceed with the inversion of the linearised KdV equation (21) for perturbations about a one-soliton solution. We assume that u_1 can be expanded in the complete set of states as

$$u_1 = \int_{-\infty}^{\infty} \mathrm{d}\lambda f(\lambda,t)\Phi^{\mathrm{A}}(x,t;\lambda) + f_1(t)\Phi_1^{\mathrm{A}}(x,t) + g_1(t)\Lambda_1^{\mathrm{A}}(x,t). \tag{39}$$

Inserting this expansion into equation (21), we find that

$$\mathcal{F} = \int_{-\infty}^{\infty} \mathrm{d}\lambda(f_t - 8\mathrm{i}\lambda^3 f)\Phi^{\mathrm{A}}(\lambda) + (f_{1t} - 8\eta^3 f_1 + 48\mathrm{i}\eta^2 g_1)\Phi_1^{\mathrm{A}} + (g_{1t} - 8\eta^3 g_1)\Lambda_1^{\mathrm{A}} \tag{40}$$

which is just an eigenfunction expansion for $\mathcal{F}$ in our basis.

Taking inner products on both sides of this equation with the adjoint states and using the orthogonality relations (36) and (37), we obtain the following first-order differential equations for the expansion coefficients:

$$f_t - 8\mathrm{i}\lambda^3 f = \frac{\langle\mathcal{F}|\Phi\rangle}{2\pi\mathrm{i}\lambda a^2(\lambda)} \tag{41}$$

$$f_{1t} - 8\eta^3 f_1 + 48\mathrm{i}\eta^2 g_1 = -2\mathrm{i}\eta\langle\mathcal{F}|\Lambda_1\rangle \tag{42}$$

$$g_{1t} - 8\eta^3 g_1 = -2\mathrm{i}\eta\langle\mathcal{F}|\Phi_1\rangle. \tag{43}$$

These equations can easily be solved, yielding

$$f(\lambda,t) = f(\lambda,0)\exp(8i\lambda^3 t) + \int_0^t dt' \frac{\langle \mathcal{F}|\Phi\rangle}{2\pi i\lambda a^2(\lambda)} \exp[8i\lambda^3(t-t')] \tag{44}$$

$$g_1(t) = g_1(0)\exp(8\eta^3 t) - 2i\eta \int_0^t dt' \langle \mathcal{F}|\Phi_1\rangle \exp[8\eta^3(t-t')] \tag{45}$$

$$\begin{aligned} f_1(t) = {} & f_1(0)\exp(8\eta^3 t) - 48i\eta^2 t g_1(0)\exp(8\eta^3 t) - 2i\eta \int_0^t dt' \langle \mathcal{F}|\Lambda_1\rangle \exp[8\eta^3(t-t')] \\ & - 96\eta^3 \int_0^t dt' \int_0^{t'} dt'' \langle \mathcal{F}|\Phi_1\rangle \exp[8\eta^3(t-t'')]. \end{aligned} \tag{46}$$

This completes the inversion of the linear operator. We note that in studying perturbations about a single soliton, where the perturbation is turned on at $t = 0$, $u = u_0$; i.e. $u_n = 0$ for $n > 1$. So, in equations (44)–(46) the initial values of the expansion coefficients are zero. However, it is possible to study perturbations of initial profiles which are close to a single soliton. Namely, at $t = 0$,

$$\begin{aligned} u(x,0) = {} & 2\eta^2 \operatorname{sech}^2 \eta x + \epsilon \int_{-\infty}^{\infty} d\lambda\, f(\lambda,0)\Phi^A(x,0;\lambda) \\ & + \epsilon f_1(0)\Phi_1^A(x,0) + \epsilon g_1(0)\Lambda_1^A(x,0) + O(\epsilon^2). \end{aligned} \tag{47}$$

The resulting correction to the soliton motion is found using the above results. Furthermore, the bound state contribution in (47) can easily be shown to be a result of small changes in the soliton amplitude and position by using equation (49) below. In the discussions below, we will assume that the initial profile is a single soliton.

3.4. Secularity conditions

Using the basis states for a one-soliton solution (see appendix A), we can rewrite the last two terms in (39),

$$B \equiv f_1\Phi_1^A + g_1\Lambda_1^A \tag{48}$$

as

$$B = \tilde{g}_1[\operatorname{sech}^2\phi + \tfrac{1}{2}\phi(\operatorname{sech}^2\phi)_\phi] + \tilde{h}_1(\operatorname{sech}^2\phi)_\phi \tag{49}$$

where the new coefficients are given by

$$\tilde{g}_1 \equiv \int_0^t dt' \langle \mathcal{F}|\operatorname{sech}^2\phi\rangle \tag{50}$$

$$\tilde{h}_1 \equiv -\frac{1}{2}\int_0^t dt' \langle \mathcal{F}|[\phi + 8\eta^3(t-t')]\operatorname{sech}^2\phi + \tanh\phi\rangle. \tag{51}$$

When $\mathcal{F}$ is independent of time, these coefficients will grow in time, unless we impose the *secularity* conditions

$$\langle \mathcal{F}|\operatorname{sech}^2\phi\rangle = 0 \tag{52}$$

$$\langle \mathcal{F}|\phi\operatorname{sech}^2\phi + \tanh\phi\rangle = 0. \tag{53}$$

Applying these conditions to the first-order equation (19) we obtain the slow time dependence of the soliton parameters [10]:

$$\eta_\tau = \frac{1}{4\eta}\int_{-\infty}^{\infty} F[u_0]\,\mathrm{sech}^2\,\phi\,d\phi \tag{54}$$

$$x_{0\tau} = -4\eta^2 \tag{55}$$

$$x_{1\tau} = \frac{1}{4\eta^3}\int_{-\infty}^{\infty} F[u_0][\phi\,\mathrm{sech}^2\,\phi + \tanh\phi]\,\mathrm{d}\phi. \tag{56}$$

The first equation determines the change in the soliton amplitude ($2\eta^2$) and width ($1/\eta$). The second of these equations gives the leading-order velocity, while the last equation will give the correction to the velocity of the soliton.

3.5. First-order correction

From this analysis one obtains the correction u_1,

$$u_1 = \int_{-\infty}^{\infty} \mathrm{d}\lambda \int_0^t \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x'\, \frac{\mathcal{F}(x',t')\Phi(x',t';\lambda)}{2\pi\mathrm{i}\lambda a^2(\lambda)}\,\exp[8\mathrm{i}\lambda^3(t-t')]\Phi^{\mathrm{A}}(x,t;\lambda). \tag{57}$$

In general, for dissipative perturbations this correction will account for the development of a decaying oscillatory tail and possibly a shelf, due to the pole at $\lambda = 0$. The height of this shelf can be determined from an analysis of u_1. Asymptotically, we have that [10, 13, 23]

$$u_1 \sim \frac{1}{4\eta^3}\int_{-\infty}^{\infty} F[u_0]\tanh^2\phi\,\mathrm{d}\phi. \tag{58}$$

A careful analysis shows that the presence of this shelf leads to an additional $\tanh^2\phi$ term in (56) [10, 13, 23]. This important effect results from using higher-order terms in the perturbation analysis. Using asymptotics, one can show that equation (56) should be replaced by [13]

$$x_{1\tau} = \frac{1}{4\eta^3}\int_{-\infty}^{\infty} F[u_0][\phi\,\mathrm{sech}^2\,\phi + \tanh\phi + \tanh^2\phi]\,\mathrm{d}\phi. \tag{59}$$

This result agrees with other results [23, 34], and has been found to be supported by numerical experiments [11].

3.6. Example: damped KdV equation

We will now turn to the analysis of the damped KdV equation, using the perturbation method presented above. In the course of this analysis we will show how the second-order results are needed to obtain equation (59) in the last section, and we will present the first-order correction.

The form of the perturbed equation, which we are using, is given as

$$u_t + 6uu_x + u_{xxx} = -\Gamma u = -\epsilon\gamma u. \tag{60}$$

We will assume an ϵ expansion in u:

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots \tag{61}$$

where u_0 is the one-soliton solution given by

$$u_0 = A\,\mathrm{sech}^2\,\phi \qquad \phi = \eta(x - \bar{x}(t)) \qquad A = 2\eta^2. \tag{62}$$

In order to obtain the slow time dependence of the amplitude and phase shift, or soliton centre, we will introduce a slow time scale and expand the soliton parameters. We assume the expansions

$$\eta = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \ldots \tag{63}$$

$$A = 2(\eta_0 + \epsilon\eta_1 + \ldots)^2 = 2\eta_0^2 + 4\epsilon\eta_0\eta_1 + \epsilon^2[2\eta_1^2 + 4\eta_0\eta_2] + \ldots \tag{64}$$

$$\bar{x} = \frac{1}{\epsilon}(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots). \tag{65}$$

Defining the two time scales, $T = t$ and $\tau = \epsilon t$, we assume that the soliton parameters depend only on τ. Rewriting the needed derivatives in the new variables (ϕ, T, τ), in (60), as

$$\begin{aligned} \partial_x &= \eta\partial_\phi \\ \partial_t &= \partial_T + \epsilon\partial_\tau + \epsilon\phi_\tau\partial_\phi + \ldots \end{aligned} \tag{66}$$

and inserting the expansion for u in the resulting expression, will give to the lowest two orders:

$$24\eta_0^5 v v_\phi - 2\eta_0^3 x_{0_\tau} v_\phi + 2\eta_0^5 v_{\phi\phi\phi} = 0 \tag{67}$$

$$\begin{aligned} &[\partial_T - \eta_0 x_{0_\tau}\partial_\phi + 6\eta_0\partial_\phi u_0 + \eta_0^3\partial_\phi^3]u_1 \\ &\quad = -4\eta_0\eta_{0_\tau} v - 120\eta_0^4\eta_1 v v_\phi - 2\eta_0\eta_{0_\tau}\phi v_\phi + 6\eta_0^2\eta_1 x_{0_\tau} v_\phi \\ &\qquad + 2\eta_0^3 x_{1_\tau} v_\phi - 10\eta_0^4\eta_1 v_{\phi\phi\phi} - 2\eta_0^2\gamma v \end{aligned} \tag{68}$$

where we have defined $v = \mathrm{sech}^2\,\phi$.

As before, the lowest-order equation has u_0 as a solution if and only if

$$x_{0_\tau} = 4\eta_0^2. \tag{69}$$

Inserting this into the first-order equation, we have

$$\mathcal{L}u_1 = (-4\eta_0\eta_{0_\tau} - 2\eta_0^2\gamma)v - 2\eta_0\eta_{0_\tau}\phi v_\phi + 2\eta_0^3 x_{1_\tau} v_\phi - 16\eta_0^4\eta_1 v_\phi. \tag{70}$$

Now, we are ready to invert this equation.

We recall that the bound state contribution to the solution will contain secular terms unless we require the secularity conditions (52) and (53). Using the driving terms in (70) the time dependence of the soliton parameters is found as

$$\eta_{0_\tau} = -\tfrac{2}{3}\gamma\eta_0 \tag{71}$$

$$x_{1_\tau} = 8\eta_0\eta_1. \tag{72}$$

This leads to the first-order correction as

$$u_1 = \int_{-\infty}^{\infty} \frac{\gamma[\exp(-8\mathrm{i}\lambda\eta_0^2 t) - \exp(8\mathrm{i}\lambda^3 t)]}{12\lambda(\mathrm{i}\lambda + \eta_0)^3(\mathrm{i}\lambda - \eta_0)\sinh(\pi\lambda/\eta_0)} \Phi^{\mathrm{A}}(\phi, t; \lambda)\,\mathrm{d}\lambda. \tag{73}$$

We now want to integrate the soliton parameter equations (71) and (72). Assuming that γ is constant, equation (71) integrates to

$$\eta_0(\tau) = \tilde{\eta}_0 \exp(-\tfrac{2}{3}\epsilon\gamma t) \qquad \tilde{\eta}_0 \equiv \eta_0(0). \tag{74}$$

Using the relation to the amplitude in (64), we have to leading order in ϵ:

$$A = 2\tilde{\eta}_0^2 \exp(-\tfrac{4}{3}\epsilon\gamma t) + \mathrm{O}(\epsilon^2). \tag{75}$$

This is the same result as has been reported in the literature.

This leaves us with the evaluation of the correction to the soliton positon. We can integrate $x_{0\tau}$ in (69) to obtain

$$x_0 = 4\int_0^{\epsilon t} \eta_0^2(s)\,\mathrm{d}s = \frac{3\tilde{\eta}_0^2}{\gamma}\left[\exp(-\tfrac{4}{3}\epsilon\gamma t) - 1\right]. \tag{76}$$

However, in order to integrate (72) for the correction to the soliton position, we need information about η_1, which can only be obtained from an analysis of the next order. Thus, for very short times, the centre of the soliton is given by (76).

From the analysis so far, we have information as to how the soliton will change under the above damping. From the multiple scale results, we see that the amplitude will decay for $\gamma > 0$, and it will grow for $\gamma < 0$. Similarly, we find from the width, $1/\eta$, that the soliton will broaden for $\gamma > 0$. The velocity of the soliton is given by

$$\frac{1}{\epsilon} x_{0\tau} = 4\tilde{\eta}_0^2 \exp(-\tfrac{4}{3}\epsilon\gamma t). \tag{77}$$

We see that for $\gamma > 0$ the soliton tends to slow down and come to a rest; however, long before this higher-order changes in the amplitude will lead to a first-order correction to the location of the soliton centre and its velocity.

In appendix B we look at the long time asymptotics of the first-order solution in (73). The result is that the first-order solution u_1 behaves asymptotically as

$$\begin{cases} \dfrac{\gamma}{6\sqrt{\pi}\,\eta_0}\left(\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right)^{-3/4} \exp\left[-\dfrac{2}{3}\left(\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right)^{3/2}\right] & \text{for } 0 < 4\eta_0^2 t < \xi \\ -\dfrac{\gamma}{3\eta_0} + \dfrac{\gamma}{6\sqrt{\pi}\,\eta_0}\left(\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right)^{-3/4} \exp\left[-\dfrac{2}{3}\left(\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right)^{3/2}\right] & \text{for } 0 < \xi < 4\eta_0^2 t \\ -\dfrac{\gamma}{3\sqrt{\pi}\,\eta_0}\left|\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right|^{-3/4} \cos\left[\dfrac{2}{3}\left|\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right|^{3/2} + \dfrac{\pi}{4}\right] & \text{for } \xi < 0 \end{cases} \tag{78}$$

for large times, and in the region where $\tanh^2\phi \simeq 1$. Here we have defined

$$\xi \equiv x - \bar{x}_0 \qquad \bar{x}_0 \equiv \bar{x} - 4\eta_0^2 t. \tag{79}$$

Thus, the major contribution comes from the region from $x = 0$ to the solitary wave position, $x = \bar{x} = \epsilon^{-1}x_0 + x_1$. This is the shelf first found numerically by Leibovich and Randall [37], and later analytically by Kaup and Newell [30]. Note that we have obtained the same height, using our solution, as they had:

$$u_1 \sim -\frac{\gamma}{3\eta_0} \qquad 0 < x - \bar{x}_0 < 4\eta_0^2 t. \tag{80}$$

We can now proceed to second order for the sole purpose of obtaining the correction needed in equation (72). Using the first-order results, the second-order equation is given by

$$\begin{aligned}\mathcal{L}u_2 = &-\tfrac{4}{3}\eta_0\eta_1\gamma v - 4\eta_0\eta_{1\tau}v + \tfrac{4}{3}\phi\eta_0\eta_1\gamma v_\phi - 2\phi\eta_0\eta_{1\tau}v_\phi \\ &+ \tfrac{2}{3}\gamma\phi u_{1\phi} - 6\eta_0 u_1 u_{1\phi} - \gamma u_1 - 36\eta_0^2\eta_1 u_1 v_\phi + 2\eta_0^3 x_{2\tau}v_\phi - u_{1\tau} \\ &- 240\eta_0^3\eta_1^2 v v_\phi - 120\eta_0^4\eta_2 v v_\phi + 6\eta_0^2\eta_1 x_{1\tau}v_\phi + 24\eta_0^3\eta_1^2 v_\phi - 20\eta_0^3\eta_1^2 v_{\phi\phi\phi} \\ &- 10\eta_0^4\eta_2 v_{\phi\phi\phi} + 24\eta_0^4\eta_2 v_\phi - 36\eta_0^2\eta_1 v u_{1\phi} + 12\eta_0^2\eta_1 u_{1\phi} - 3\eta_0^2\eta_1 u_{1\phi\phi\phi}.\end{aligned} \tag{81}$$

To obtain the time dependence of η_1, we need to employ the condition (52). Since the resulting expressions become complicated and uninformative, we will use the large time behaviour of the first-order solution. We will assume

$$u_1 \sim -\frac{\gamma}{3\eta_0}\theta(-\phi) \qquad u_{1\phi} \simeq 0 \tag{82}$$

where $\theta(x)$ is the Heaviside step function and the condition on v is consistent with the assumptions made in obtaining the asymptotic form for u_1. In doing this we find that

$$\eta_1 \sim -\frac{\gamma}{24\eta_0^2}. \tag{83}$$

Inserting this asymptotic result into equation (72), we find

$$x_{1\tau} \sim -\frac{\gamma}{3\eta_0}. \tag{84}$$

This is the same result as is obtained using equation (59).

For completeness, we further integrate this equation and add it to the fully integrated form for the soliton centre in (76):

$$\bar{x} \simeq \frac{1}{\epsilon}x_0 + x_1 = \frac{3\bar{\eta}_0^2}{\gamma}\left[\exp(-\tfrac{4}{3}\epsilon\gamma t) - 1\right] + \frac{1}{2\bar{\eta}_0}\left[1 - \exp(\tfrac{2}{3}\epsilon\gamma t)\right]. \tag{85}$$

However, we must remember that this result relies on the approximation of large times.

The above analysis can be generalised to arbitrary perturbations. Using the asymptotic form of u_1, which is given in equation (58), we find that the correction to the soliton position is provided by equation (59) [10, 13]. Again, this is an asymptotic result, but holds fairly well for dissipative perturbations. Karpman and Maslov have obtained this same correction [24], though the approximation that they used did not clearly show that the additional term in (59) was due to higher-order terms in the perturbation analysis.

4. The perturbed NLS equation

4.1. Perturbation expansion

We now turn to the perturbed nonlinear Schrödinger equation (NLS) of the form

$$\mathrm{i}q_t - q_{xx} - 2|q|^2 q = \epsilon F. \tag{86}$$

As before we look for solutions of this equation which are close to the one-soliton solution

$$q_0 = -2\mathrm{i}\eta \exp[\mathrm{i}(\beta - \xi\theta/\eta)] \operatorname{sech}\theta \qquad \theta = 2\eta(x - \bar{x}) \tag{87}$$

where, to leading order,

$$\beta_t = -4(\xi^2 + \eta^2) \qquad \bar{x}_t = 4\xi. \tag{88}$$

In order to find such solutions, we introduce the expansions:

$$q = q_0 + \epsilon q_1 \tag{89}$$

$$\beta = \frac{1}{\epsilon}(\beta_0 + \epsilon\beta_1) \tag{90}$$

$$\bar{x} = \frac{1}{\epsilon} x_0 + x_1 \tag{91}$$

and assume that the soliton parameters $(\eta, \beta, \xi, \bar{x})$ depend on a slow time scale $\tau = \epsilon t$.

Inserting this information into equation (86), we find to leading order that q_0 satisfies the NLS equation provided

$$x_{0\tau} = 4\xi \qquad \beta_{0\tau} = -4(\xi^2 + \eta^2). \tag{92}$$

The first-order equation we have to solve is

$$\mathrm{i}q_{1t} - q_{1xx} - 2q_0^2 q_1^* - 4q_0 q_0^* q_1 = F_1 - S_1 \equiv P_1 \tag{93}$$

where F_1 is the leading-order contribution from the perturbation term in (86) and S_1 is the contribution from the slow time scales:

$$S_1 = [(4\mathrm{i}\eta\xi x_{1\tau} + 2\mathrm{i}\eta\beta_{1\tau} + 2\eta_\tau)v - 2\mathrm{i}\xi_\tau\theta v - 4\eta^2 x_{1\tau} v_\theta + 2\eta_\tau \theta v_\theta]\mathrm{e}^{-\mathrm{i}\delta} \tag{94}$$

where

$$v \equiv \operatorname{sech}\theta \qquad \delta \equiv \xi\theta/\eta - \beta. \tag{95}$$

The aim is to solve equation (93) for q_1; however, this equation also involves q_1^*. By using the complex conjugate of equation (93) we can rewrite it as a linear operator operating on a two-dimensional vector. Namely, we can write the equation as

$$\mathcal{L}\begin{pmatrix} q_1 \\ q_1^* \end{pmatrix} = \begin{pmatrix} P_1 \\ -P_1^* \end{pmatrix} \tag{96}$$

where the linear operator is given by

$$\mathcal{L} = \begin{pmatrix} \mathrm{i}\partial_t - \partial_x^2 - 4q_0q_0^* & -2q_0^2 \\ 2q_0^{*2} & \mathrm{i}\partial_t + \partial_x^2 + 4q_0q_0^* \end{pmatrix}. \tag{97}$$

In order to find the correction to the soliton we need to invert the linear operator. In the next section we show how this can be done using some tools from inverse scattering theory.

4.2. Perturbation basis

It is well known that the nonlinear Schrödinger equation is a compatibility condition for the Lax pair [3]

$$v_x = \mathbf{L}v = \begin{pmatrix} -\mathrm{i}\lambda & q \\ -q^* & \mathrm{i}\lambda \end{pmatrix} v \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{98}$$

$$v_t = \mathbf{M}v = \begin{pmatrix} 2\mathrm{i}\lambda^2 - \mathrm{i}qq^* & -2\lambda q - \mathrm{i}q_x \\ 2\lambda q^* - \mathrm{i}q^*_x & -2\mathrm{i}\lambda^2 + \mathrm{i}qq^* \end{pmatrix} v. \tag{99}$$

Namely, one obtains from $v_{xt} = v_{tx}$ with $\lambda_t = 0$ the NLS equation

$$\mathrm{i}q_t - q_{xx} - 2|q|^2 q = 0. \tag{100}$$

Using these equations, one finds that

$$(\mathrm{i}\partial_t - \partial_x^2)v_1^2 = 4qq^* v_1^2 - 2q^2 v_2^2 \tag{101}$$

$$(\mathrm{i}\partial_t + \partial_x^2)v_2^2 = -4qq^* v_2^2 + 2q^{*2} v_1^2. \tag{102}$$

From the form of $\mathcal{L}$ we have that

$$\mathcal{L}\begin{pmatrix} v_1^2 \\ -v_2^2 \end{pmatrix} = 0. \tag{103}$$

Therefore, we have found the homogeneous solutions to our first-order equation. In principal we can now use variation of parameters to solve for q_1.

Implementing this scheme involves solving for v in equations (98) and (99), given the q in equation (87). Following Ablowitz and Segur [3], we find two linearly independent solutions of (98):

$$\psi(x,t;\lambda) = \begin{pmatrix} \dfrac{-\eta}{\lambda - \zeta^*} \exp[\mathrm{i}(\lambda x + \beta - \xi\theta/\eta)] \operatorname{sech}\theta \\ \dfrac{1}{\lambda - \zeta^*} \exp(\mathrm{i}\lambda x)[\lambda - \xi + \mathrm{i}\eta \tanh\theta] \end{pmatrix} \tag{104}$$

$$\bar{\psi}(x,t;\lambda) = \begin{pmatrix} \dfrac{1}{\lambda - \zeta} \exp(-\mathrm{i}\lambda x)[\lambda - \xi - \mathrm{i}\eta \tanh\theta] \\ \dfrac{\eta}{\lambda - \zeta} \exp[-\mathrm{i}(\lambda x + \beta - \xi\theta/\eta)] \operatorname{sech}\theta \end{pmatrix} \tag{105}$$

where

$$\zeta \equiv \xi + \mathrm{i}\eta. \tag{106}$$

However, these forms do not satisfy equation (99). Assuming from (88) that

$$\theta_t = -8\eta\xi \qquad \beta_t = -4(\xi^2 + \eta^2) \tag{107}$$

we find that

$$\psi_t - \mathbf{M}\psi = 2\mathrm{i}\lambda^2\psi. \tag{108}$$

We could easily adjust ψ to satisfy (99), however, by using the form in (104), we find that equation (103) is modified as

$$\mathcal{L}\begin{pmatrix} \psi_1^2 \\ -\psi_2^2 \end{pmatrix} = -4\lambda^2 \begin{pmatrix} \psi_1^2 \\ -\psi_2^2 \end{pmatrix}. \tag{109}$$

This is an eigenvalue problem for our operator, with eigenvalues $-4\lambda^2$ and eigenfunctions $\Omega \equiv \binom{\psi_1^2}{-\psi_2^2}$.

In summary, we have related the solutions of the associated spectral problem for the NLS to the eigenfunctions of our linear operator. The Jost solutions $\psi(x,t;\lambda)$, $\bar{\psi}(x,t;\lambda)$, which are given in equations (104) and (105), are related to the eigenfunction of our operator by the eigenvalue problems:

$$\mathcal{L}\Omega = -4\lambda^2\Omega \qquad \Omega = \begin{pmatrix} \psi_1^2 \\ -\psi_2^2 \end{pmatrix} \tag{110}$$

$$\mathcal{L}\bar{\Omega} = 4\lambda^2\bar{\Omega} \qquad \bar{\Omega} = \begin{pmatrix} \bar{\psi}_1^2 \\ -\bar{\psi}_2^2 \end{pmatrix}. \tag{111}$$

In the next section we would like to use this basis of eigenfunctions to solve for the first-order correction in the perturbation theory from equation (96). We will assume that $\binom{q_1}{q_1^*}$ can be expanded in this basis and solve for the unknown expansion coefficients. However, the basis $\Omega, \bar{\Omega}$ is not complete. Kaup has provided the tools needed to complete the basis [27]. Namely, we will need to add the discrete states

$$\Omega_1(x,t) = \Omega(x,t;\xi + \mathrm{i}\eta) \qquad \Lambda_1(x,t) = \partial_\lambda \Omega(x,t;\xi + \mathrm{i}\eta) \tag{112}$$

$$\bar{\Omega}_1(x,t) = \bar{\Omega}(x,t;\xi - \mathrm{i}\eta) \qquad \bar{\Lambda}_1(x,t) = \partial_\lambda \bar{\Omega}(x,t;\xi - \mathrm{i}\eta). \tag{113}$$

For these states the action of the linear operator is found to be

$$\mathcal{L}\Omega_1 = -4\lambda_1^2\Omega_1 \qquad \lambda_1 = \xi + \mathrm{i}\eta \tag{114}$$

$$\mathcal{L}\Lambda_1 = -4\lambda_1^2\Lambda_1 - 8\lambda_1\Omega_1 \tag{115}$$

$$\mathcal{L}\bar{\Omega}_1 = 4\lambda_1^{*2}\bar{\Omega}_1 \tag{116}$$

$$\mathcal{L}\bar{\Lambda}_1 = 4\lambda_1^{*2}\bar{\Lambda}_1 - 8\lambda_1^*\bar{\Omega}_1. \tag{117}$$

Also, associated with this basis are the adjoint states:

$$\Omega^{\mathrm{A}}(x,t;\lambda) = (\phi_2^2, \phi_1^2) \qquad \bar{\Omega}^{\mathrm{A}}(x,t';\lambda) = (\bar{\phi}_2^2, \bar{\phi}_1^2) \tag{118}$$

and their associated discrete states, which are defined like those in equations (112) and (113). Here ϕ and $\bar{\phi}$ are the following solutions to equation (98):

$$\phi(x,t;\lambda) = \begin{pmatrix} \dfrac{1}{\lambda - \zeta^*} \exp(-\mathrm{i}\lambda x)[\lambda - \xi - \mathrm{i}\eta \tanh\theta] \\ \dfrac{\eta}{\lambda - \zeta^*} \exp[-\mathrm{i}(\lambda x + \beta - \xi\theta/\eta)] \operatorname{sech}\theta \end{pmatrix} \tag{119}$$

$$\bar{\phi}(x,t;\lambda) = \begin{pmatrix} \dfrac{\eta}{\lambda - \zeta} \exp[\mathrm{i}(\lambda x + \beta - \xi\theta/\eta)] \operatorname{sech}\theta \\ \dfrac{-1}{\lambda - \zeta} \exp(\mathrm{i}\lambda x)[\lambda - \xi + \mathrm{i}\eta \tanh\theta] \end{pmatrix}. \tag{120}$$

Finally, we will need the orthogonality relations between the basis and adjoint states for the inversion of the linear operator. These are [27]

$$\langle\Omega(\lambda')|\Omega^{A}(\lambda)\rangle = -\pi a^2(\lambda)\delta(\lambda-\lambda') \tag{121}$$

$$\langle\bar{\Omega}(\lambda')|\bar{\Omega}^{A}(\lambda)\rangle = \pi \bar{a}^2(\lambda)\delta(\lambda-\lambda') \tag{122}$$

$$\begin{aligned}\langle\Omega_1|\Lambda_1^{A}\rangle &= \langle\Lambda_1|\Omega_1^{A}\rangle = -\tfrac{1}{2}(a_1')^2\\ \langle\Lambda_1|\Lambda_1^{A}\rangle &= -\tfrac{1}{2}\mathrm{i}a''a_1'\end{aligned} \tag{123}$$

$$\begin{aligned}\langle\bar{\Omega}_1|\bar{\Lambda}_1^{A}\rangle &= \langle\bar{\Lambda}_1|\bar{\Omega}_1^{A}\rangle = -\tfrac{1}{2}(\bar{a}_1')^2\\ \langle\bar{\Lambda}_1|\bar{\Lambda}_1^{A}\rangle &= -\tfrac{1}{2}\mathrm{i}\bar{a}''\bar{a}_1'\end{aligned} \tag{124}$$

where

$$\begin{aligned}a(\lambda) &= \frac{\lambda-\zeta}{\lambda-\zeta^*} \qquad \bar{a}(\lambda) = \frac{\lambda-\zeta^*}{\lambda-\zeta}\\ a_1' &= \left.\frac{\mathrm{d}a}{\mathrm{d}\lambda}\right|_{\lambda_1} \qquad \bar{a}_1' = \left.\frac{\mathrm{d}\bar{a}}{\mathrm{d}\lambda}\right|_{\lambda_1^*}.\end{aligned} \tag{125}$$

4.3. Inversion of the linear operator

We are now in a position to invert our linear operator and solve for the first-order correction to the solution of the perturbed NLS equation. We want to solve the general equation (96), which we write as

$$\mathcal{L}Q = \mathcal{F}. \tag{126}$$

We assume that Q can be expanded in our basis as

$$Q = \int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda}{\pi}[f(\lambda,t)\Omega(\lambda) + \bar{f}(\lambda,t)\bar{\Omega}] + f_1(t)\Omega_1 + g_1(t)\Lambda_1 + \bar{f}_1(t)\bar{\Omega}_1 + \bar{g}_1(t)\bar{\Lambda}_1. \tag{127}$$

Now apply $\mathcal{L}$ to this, using the equations from the previous section for the operator acting on the basis states. This results in

$$\begin{aligned}\mathcal{F} = \mathcal{L}Q = \int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda}{\pi}[(\mathrm{i}f_t - 4\lambda^2 f)\Omega + (\mathrm{i}\bar{f}_t + 4\lambda^2\bar{f})\bar{\Omega}] + (\mathrm{i}f_{1t} - 4\lambda_1^2 f_1 - 8\lambda_1 g_1)\Omega_1\\ + (\mathrm{i}g_{1t} - 4\lambda_1^2 g_1)\Lambda_1 + (\mathrm{i}\bar{f}_{1t} + 4\lambda_1^{*2}\bar{f}_1 + 8\lambda_1^*\bar{g}_1)\bar{\Omega}_1 + (\mathrm{i}\bar{g}_{1t} + 4\lambda_1^{*2}\bar{g}_1)\bar{\Lambda}_1.\end{aligned} \tag{128}$$

This is just another expansion in the basis. We can use the orthogonality products to pull out the coefficients in this equation. Doing this leads to the first-order equations:

$$f_t + 4\mathrm{i}\lambda^2 f = \frac{\mathrm{i}\langle\mathcal{F}|\Omega^{A}\rangle}{a^2(\lambda)} \tag{129}$$

$$\bar{f}_t - 4\mathrm{i}\lambda^2 \bar{f} = -\frac{\mathrm{i}\langle \mathcal{F}|\bar{\Omega}^{\mathrm{A}}\rangle}{\bar{a}^2(\lambda)} \tag{130}$$

$$g_{1t} + 4\mathrm{i}\lambda_1^2 g_1 = \frac{2}{(a_1')^2}\langle \mathcal{F}|\Omega_1^{\mathrm{A}}\rangle \tag{131}$$

$$\bar{g}_{1t} - 4\mathrm{i}\lambda_1^{*2}\bar{g}_1 = \frac{2}{(\bar{a}_1')^2}\langle \mathcal{F}|\bar{\Omega}_1^{\mathrm{A}}\rangle \tag{132}$$

$$f_{1t} + 4\mathrm{i}\lambda_1^2 f_1 - 8\lambda_1 g_1 = \frac{2}{(a_1')^2}\langle \mathcal{F}|\Omega_1^{\mathrm{A}}\rangle - \frac{2a_1''}{(a_1')^3}\langle \mathcal{F}|\Omega_1^{\mathrm{A}}\rangle \tag{133}$$

$$\bar{f}_{1t} - 4\mathrm{i}\lambda_1^{*2}\bar{f}_1 + 8\lambda_1^*\bar{g}_1 = \frac{2}{(\bar{a}_1')^2}\langle \mathcal{F}|\bar{\Omega}_1^{\mathrm{A}}\rangle - \frac{2\bar{a}_1''}{(\bar{a}_1')^3}\langle \mathcal{F}|\bar{\Omega}_1^{\mathrm{A}}\rangle. \tag{134}$$

These equations can be integrated to find the expansion coefficients, leading to the solution of equation (126). Therefore, we have inverted the linear operator and can return to perturbation theory.

4.4. Secularity conditions and the soliton parameters

We have a method for solving for the first-order correction to the perturbation problem. A careful analysis of the discrete contribution in equation (127) will show that these terms will grow in time unless we force $f_1 = 0$, $g_1 = 0$, $\bar{f}_1 = 0$ and $\bar{g}_1 = 0$. This leads to the conditions

$$\langle \mathcal{F}|\Omega_1^{\mathrm{A}}\rangle = -\langle \mathcal{F}|\bar{\Omega}_1^{\mathrm{A}}\rangle^* = 0 \tag{135}$$

$$\langle \mathcal{F}|\Lambda_1^{\mathrm{A}}\rangle = -\langle \mathcal{F}|\bar{\Lambda}_1^{\mathrm{A}}\rangle^* = 0. \tag{136}$$

These secularity conditions can be applied to the pertubation $P_1 = F_1 - S_1$ in equation (93). Computing the inner products in (136) with $\mathcal{F} = P_1$, and using the definitions of the adjoint states, we find that the secularity conditions lead to equations for the slow time dependence of the soliton parameters:

$$\eta_\tau = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\theta\, \mathrm{Re}[F_1 \mathrm{e}^{\mathrm{i}\delta}]w \tag{137}$$

$$\xi_\tau = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\theta\, \mathrm{Im}[F_1 \mathrm{e}^{\mathrm{i}\delta}]w_\theta \tag{138}$$

$$x_{1\tau} = \frac{1}{4\eta^2}\int_{-\infty}^{\infty} \mathrm{d}\theta\, \mathrm{Re}[F_1 \mathrm{e}^{\mathrm{i}\delta}]\theta w \tag{139}$$

$$\beta_{1\tau} = \frac{1}{2\eta}\int_{-\infty}^{\infty} \mathrm{d}\theta\, \mathrm{Im}[F_1 \mathrm{e}^{\mathrm{i}\delta}](\theta w_\theta + w) - 2\xi x_{1\tau} \tag{140}$$

where we have defined $w \equiv \operatorname{sech}\theta$. We note that these are the same results as given by Karpman [22].

4.5. First-order correction

In the previous section we have eliminated the bound state correction to the perturbed solution. The first-order correction is now

$$\begin{pmatrix} q_1 \\ q_1^* \end{pmatrix} = \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{\pi}[f(\lambda,t)\Omega(\lambda) + \bar{f}(\lambda,t)\bar{\Omega}] \tag{141}$$

where the expansion coefficients are obtained from equations (129) and (130) as

$$f(\lambda,t) = \frac{\mathrm{i}}{a^2}\int_0^t \langle P_1|\Omega^{\mathrm{A}}\rangle \exp[-4\mathrm{i}\lambda^2(t-t')]\,\mathrm{d}t' \tag{142}$$

$$\bar{f}(\lambda,t) = -\frac{\mathrm{i}}{\bar{a}^2}\int_0^t \langle P_1|\bar{\Omega}^{\mathrm{A}}\rangle \exp[4\mathrm{i}\lambda^2(t-t')]\,\mathrm{d}t'. \tag{143}$$

Recalling that $P_1 = F_1 - S_1$, we find that S_1 does not contribute to the coefficients in (142) and (143). Therefore, we can replace the P_1 by F_1 in these equations. Using equation (141), and the forms for the basis and adjoint functions, we can write a formula for q_1. After some work, we have

$$\begin{aligned} q_1 = {} & \frac{\mathrm{e}^{-\mathrm{i}\delta}}{8\pi\eta^4}\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu\,\mathrm{e}^{\mathrm{i}\mu\theta}}{(\mu^2+1)^3}\{\mu^2 I(\mu)^*(\mu+\mathrm{i}\tanh\theta)^2 - I(-\mu)\,\mathrm{sech}^2\,\theta\} \\ & - \frac{\mathrm{e}^{-\mathrm{i}\delta}}{8\pi\eta^4}\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu\,\mathrm{e}^{\mathrm{i}\mu\theta}}{(\mu^2+1)^3}\{\mu^2 I(\mu)^* \exp[4\mathrm{i}\eta^2 t(\mu^2+1)](\mu+\mathrm{i}\tanh\theta)^2 \\ & - I(-\mu)\exp[-4\mathrm{i}\eta^2 t(\mu^2+1)]\,\mathrm{sech}^2\,\theta\} \end{aligned} \tag{144}$$

where

$$I(\mu) \equiv \int_{-\infty}^{\infty} \mathrm{d}\theta[(F_1\,\mathrm{e}^{\mathrm{i}\delta})^*(\mu+\mathrm{i}\tanh\theta)^2 - (F_1\,\mathrm{e}^{\mathrm{i}\delta})\,\mathrm{sech}^2\,\theta]\,\mathrm{e}^{\mathrm{i}\mu\theta}. \tag{145}$$

In equation (144) the first term can be responsible for additional shifts in the modulus and velocity of the solution of the perturbed problem. The second term is a *radiation* term, which will acount for oscillations leaving from the soliton centre, as well as oscillations at the peak of the solution. We note that there is no need to obtain corrections to equations (139) and (140) in the same manner as we had done for the KdV equation, since a shelf does not form in the perturbed NLS solution. This is because there is no $\mu = 0$ pole in the integrands in (144).

5. The perturbed sine-Gordon equation

5.1. Perturbation expansion

We finally demonstrate this method for the perturbed sine-Gordon equation in light-cone coordinates, which is given by

$$u_{xt} - \sin u = \epsilon F. \tag{146}$$

We begin by expanding u as

$$u = u_0 + \epsilon u_1 + \dots \tag{147}$$

about about a single-soliton solution; in particular, we use the kink solution:

$$u_0 = 4\tan^{-1} e^{\phi} \qquad \phi = 2\eta\left(x - \frac{1}{\epsilon}x_0 - x_1\right). \tag{148}$$

and we allow the soliton parameters to depend on a slow time scale $\tau = \epsilon t$. Inserting this information into the perturbed equation (146), we find that the leading-order velocity is

$$x_{0\tau} = -\frac{1}{4\eta^2} \tag{149}$$

and the first-order equation, which we need to solve, is given by

$$\mathcal{L}u_1 = F_1 - 4\eta_\tau w - 4\eta_\tau \phi w_\phi + 8\eta^2 x_{1\tau} w_\phi \equiv \mathcal{F} \tag{150}$$

where

$$\mathcal{L} = \partial xt - \cos u_0 \qquad w \boxminus \operatorname{sech}\phi. \tag{151}$$

5.2. Perturbation basis

As in the other cases, we need the Lax pair [1]:

$$v_x = \mathbf{L}v \qquad \mathbf{L} = \begin{pmatrix} -\mathrm{i}\lambda & q \\ -q & \mathrm{i}\lambda \end{pmatrix} \qquad q = -\tfrac{1}{2}u_x \tag{152}$$

$$v_t = \mathbf{M}v \qquad \mathbf{M} = \begin{pmatrix} (\mathrm{i}/4\lambda)\cos u & (\mathrm{i}/4\lambda)\sin u \\ (\mathrm{i}/4\lambda)\sin u & -(\mathrm{i}/4\lambda)\cos u \end{pmatrix}. \tag{153}$$

Cross differentiating, $v_{xt} = v_{tx}$, and requiring $\lambda_t = 0$, we obtain the unperturbed equation ($\epsilon = 0$ in (146)).

Consider the squared states,

$$\begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix}.$$

Differentiating with respect to t, equations for $(v_1^2)_t$ and $(v_2^2)_t$ can be obtained. Adding and subtracting the resulting equations, we find the states which satisfy the homogeneous version of equation (146). Denoting the states by Ω, and the linear operator by $\mathcal{L}$, we have

$$\Omega = v_1^2 + v_2^2 \qquad \mathcal{L} = \partial_{xt} - \cos u_0. \tag{154}$$

For these cases, $\mathcal{L}\Omega = 0$.

We see that the appropriate independent states in which to expand the first-order solution, u_1, of this equation are of the form $\Omega = v_1^2 + v_2^2$. As discussed in

appendix A, Kaup has shown that we need to include the bound states $\Omega_1 = \Omega(\lambda_1)$ and $\Lambda_1 = \partial_\lambda \Omega(\lambda_1)$ to form a complete basis. Depending on whether $v = \phi$, or $v = \psi$, the standard Jost solutions [3], we have two possible basis sets:
(A)

$$\Omega = \phi_1^2 + \phi_2^2 = \frac{\mathrm{e}^{-\mathrm{i}\beta}}{(\mathrm{i}\lambda - \eta)^2}[\eta^2 - \lambda^2 + 2\mathrm{i}\lambda\eta \tanh\phi]$$

$$\Omega_1 = \frac{1}{2}\exp(-\phi_0)\operatorname{sech}\phi$$

$$\Lambda_1 = -\frac{\mathrm{i}}{2\eta}\exp(-\phi_0)\left(\phi - \phi_0 - \frac{1}{\eta}t\right)\operatorname{sech}\phi \tag{155}$$

(B)

$$\Omega = \psi_1^2 + \psi_2^2 = \frac{\mathrm{e}^{\mathrm{i}\beta}}{(\mathrm{i}\lambda - \eta)^2}[\eta^2 - \lambda^2 - 2\mathrm{i}\lambda\eta \tanh\phi]$$

$$\Omega_1 = \frac{1}{2}\exp(\phi_0)\operatorname{sech}\phi$$

$$\Lambda_1 = +\frac{\mathrm{i}}{2\eta}\exp(\phi_0)\left(\phi - \phi_0 - \frac{1}{\eta}t\right)\operatorname{sech}\phi \tag{156}$$

where

$$\phi_0 = \frac{1}{\epsilon}x_0 + x_1 - \frac{t}{2\eta} \qquad \beta = 2\lambda x - \frac{t}{2\lambda}. \tag{157}$$

We will use basis set (A) in what follows.

5.3. Inversion of the linear operator

Now, assume that u_1 can be written as a linear combination of Ω:

$$u_1 = \int f(\lambda, t)\Omega(\lambda)\,\mathrm{d}\lambda \tag{158}$$

where a possible sum over the discrete spectrum is suppressed for now. We operate on u_1 with $\mathcal{L}$ to obtain

$$\mathcal{F} = \mathcal{L}u_1 = \int f_t \Omega_x \,\mathrm{d}\lambda. \tag{159}$$

In this case we do not get Ω in the integrand, as we had seen earlier for the KdV and NLS perturbations. However, we can differentiate $\Omega = (v_1^2 + v_2^2)$ with respect to x, and use (152) to find

$$\Omega_x = 2\mathrm{i}\lambda(v_2^2 - v_1^2). \tag{160}$$

Writing $\hat{\Omega} = (v_2^2 - v_1^2)$, we now have for (159):

$$\mathcal{F} = \int 2\mathrm{i}\lambda f_t \hat{\Omega}\,\mathrm{d}\lambda. \tag{161}$$

Now the f_t can be extracted from the integral by using the adjoint state associated with $\hat{\Omega}$ and the respective orthogonality conditions.

Using the above ideas, we now assume that

$$u_1 = \int f(\lambda, t)\Omega(\lambda)\,\mathrm{d}\lambda + f_1(t)\Omega_1 + g_1(t)\Lambda_1. \tag{162}$$

Inserting this into equation (150) yields

$$\mathcal{F} = \int 2\mathrm{i}\lambda f_t \hat{\Omega}\,\mathrm{d}\lambda + [2\mathrm{i}\lambda_1 f_{1t} + 2\mathrm{i}g_{1t}]\hat{\Omega}_1 + 2\mathrm{i}\lambda_1 g_{1t}\hat{\Lambda}_1. \tag{163}$$

Here we have used (160) and have defined

$$\hat{\Omega} = \phi_2^2 - \phi_1^2 \qquad \hat{\Omega}_1 = \hat{\Omega}(\lambda_1) \qquad \hat{\Lambda}_1 = \partial_\lambda \hat{\Omega}(\lambda_1). \tag{164}$$

In order to extract the expansion coefficients from equation (163), we need the adjoint states to (164), which are given by

$$\hat{\Omega}^{\mathrm{A}} = \psi_1^2 + \psi_2^2 \qquad \hat{\Omega}_1^{\mathrm{A}} = \hat{\Omega}^{\mathrm{A}}(\lambda_1) \qquad \hat{\Lambda}_1^{\mathrm{A}} = \partial_\lambda \hat{\Omega}^{\mathrm{A}}(\lambda_1). \tag{165}$$

These adjoint states are just the states we had provided in (156).

The orthogonality relations can be obtained from those derived by Kaup in [27], which we discuss in appendix A:

$$\begin{aligned} &\langle \hat{\Omega}^{\mathrm{A}}(\lambda') | \hat{\Omega}(\lambda) \rangle = -\pi a^2(\lambda)\delta(\lambda - \lambda') \\ &\langle \hat{\Omega}_1^{\mathrm{A}} | \hat{\Lambda}_1 \rangle = \langle \hat{\Lambda}_1^{\mathrm{A}} | \hat{\Omega}_1 \rangle = \frac{\mathrm{i}}{4\eta^2} \\ &\langle \hat{\Lambda}_1^{\mathrm{A}} | \hat{\Lambda}_1 \rangle = 0. \end{aligned} \tag{166}$$

Multiplying (163) by the adjoint states and integrating, we obtain first-order equations for the expansion coefficients. Solving these equations with the initial conditions given at $t = 0$, we find

$$f(\lambda, t) = f(\lambda, 0) - \int_0^t \frac{\langle \hat{\Omega} | \mathcal{F} \rangle}{2\pi\mathrm{i}\lambda a^2(\lambda)}\,\mathrm{d}t' \tag{167}$$

$$f_1(t) = f_1(0) - 2\int_0^t [\mathrm{i}\eta \langle \hat{\Lambda}_1^{\mathrm{A}} | \mathcal{F} \rangle - \langle \hat{\Omega}_1^{\mathrm{A}} | \mathcal{F} \rangle]\,\mathrm{d}t' \tag{168}$$

$$g_1(t) = g_1(0) - 2\mathrm{i}\eta \int_0^t \langle \hat{\Omega}_1^{\mathrm{A}} | \mathcal{F} \rangle\,\mathrm{d}t'. \tag{169}$$

5.4. Secularity conditions

From equations (156) and (162) we can write the solution in the form

$$u_1(x,t) = \int_{-\infty}^{\infty} \mathrm{d}\lambda\, f(\lambda,t)\Omega(x,,t;\lambda) + B(x,t) \tag{170}$$

where we have defined the bound state contribution

$$\begin{aligned} B(x,t) &= f_1\Omega_1 + g_1\Lambda_1 \\ &= \frac{1}{2}\exp(-\phi_0)\left[f_1 \operatorname{sech}\phi - \mathrm{i}g_1\left(2x - \frac{1}{2\eta^2}t\right)\operatorname{sech}\phi\right]. \end{aligned} \tag{171}$$

From equations (168) and (169) we see that f_1 and g_1 can grow in time. We can get rid of any growth in time in these bound state terms by requiring

$$g_1 \equiv 0 \Rightarrow \langle \mathcal{F} | \operatorname{sech}\phi\rangle = 0 \tag{172}$$

$$f_1 \equiv 0 \Rightarrow \langle \mathcal{F} | x \operatorname{sech}\phi\rangle = 0. \tag{173}$$

Inserting $\mathcal{F}$ from equation (150), we find the resulting expressions

$$\eta_\tau = \frac{1}{4}\int_{-\infty}^{\infty} F_1 \operatorname{sech}\phi\, \mathrm{d}\phi \tag{174}$$

$$x_{1\tau} = \frac{1}{4}\int_{-\infty}^{\infty} F_1\phi \operatorname{sech}\phi\, \mathrm{d}\phi. \tag{175}$$

5.5. First-order correction

The resulting first-order solution is now given by

$$u(x,t) = \int_{\infty}^{\infty} \mathrm{d}\lambda \left[f(\lambda,0) - \int_0^t \frac{\mathrm{d}t'}{2\pi\mathrm{i}\lambda a^2(\lambda)}\langle \psi_1^2 + \psi_2^2 | F_1\rangle\right][\phi_1^2(x,t) + \phi_2^2(x,t)]. \tag{176}$$

Finally, we note that we can rewrite this equation in the form

$$u_1(x,t) = \int_{-\infty}^{\infty} \mathrm{d}\lambda\, f(\lambda,0)\Omega(\lambda) + \int_0^t \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x'\, F(x',t')G(x,t;x',t') \tag{177}$$

where

$$G(x,t;x',t') \equiv \frac{1}{2\pi\mathrm{i}\lambda a^2(\lambda)}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \hat{\Omega}^{\mathrm{A}}(x',t';\lambda)\Omega(x,t;\lambda). \tag{178}$$

Here $G(x,t;x',t')$ is the Green function for the problem

$$\begin{aligned} &\mathcal{L}G(x,t;x',t') = 0 \qquad t' > t \\ &\lim_{t'\to t} G(x,t;x',t') = \delta(x-x'). \end{aligned} \tag{179}$$

This problem was solved by Keener and McLaughlin [31], who had obtained the same Green function (178).

6. Discussion

In this paper we have presented a direct approach to the study of soliton perturbations. In the three cases examined, we have obtained the usual results for the effects of the perturbation on the soliton shape and position. We have also obtained the first-order correction.

In examining the damped KdV soliton, we have shown how the appearance of a shelf can affect the soliton position. The usual result for this effect is shown to be produced actually by higher-order terms in the perturbation analysis. In examining this effect for the damped KdV equation, we have seen that it is possible, in principle, to push the analysis to obtain second-order results.

This method has been used to study other perturbations such as the damped KdV equations of Ott and Sudan [42], Hamiltonian perturbations of the KdV [10], and the damped stochastic KdV equation for spatially dependent noise [10, 12]. Recently, the effects of truncation error in discretisations of the KdV and NLS equations were also studied [14–16]. These errors can be treated as perturbations of the respective equations, and the perturbation method can be used in discussions of the usefulness of such perturbations, as well as for investigating the first-order oscillations which may result. A further analysis of the first-order corrections will be published elsewhere.

This method can be used to study perturbations of other integrable nonlinear evolution equations, which have not been examined to date. We are looking into perturbations of loop solitons [36], the coupled nonlinear Schrödinger equation [38], and some two-dimensional equations, such as the Kadomtsev–Petviashvili [21]. The perturbation basis for these equations are easily found through the connection of the Lax pair, the recursion operator and the linearised evolution operator [45].

Acknowledgments

The author would like to thank Dr D J Kaup introducing me to this subject and for his partial support of this study through NSF grant DMS-8501325 and grant AFSOR-86-0277.

Appendix A. One-soliton solution of the Schrödinger eigenvalue problem

In order to compute the appropriate eigenfunctions for the basis of the perturbation expansion of u_1 in section 3, we need to solve the equations

$$v_{xx} + (\lambda^2 + q)v = 0 \tag{A1}$$

$$v_t + v_{xxx} + 3(q - \lambda^2)v_x = \gamma v \tag{A2}$$

for the Jost functions ϕ_2, ψ_2, which satisfy the boundary conditions

$$\phi_2 \sim \begin{cases} \exp[-\mathrm{i}\lambda(x - x_0)] & x \to -\infty \\ a\exp[-\mathrm{i}\lambda(x - x_0)] + b\exp[\mathrm{i}\lambda(x - x_0)] & x \to +\infty \end{cases} \tag{A3}$$

and

$$\psi_2 \sim \begin{cases} \exp[\mathrm{i}\lambda(x - x_0)] & x \to +\infty \\ a\exp[\mathrm{i}\lambda(x - x_0)] - b^*\exp[-\mathrm{i}\lambda(x - x_0)] & x \to -\infty. \end{cases} \tag{A4}$$

We consider the one-soliton case

$$q = 2\eta^2 \operatorname{sech}^2 \eta(x - 4\eta^2 t - x_0) = 2\eta^2 \operatorname{sech}^2 \phi. \tag{A5}$$

The general solution to the Schrödinger equation (A1), using this q, is given by [7]

$$v(x,t) = A_1(t)[\eta \tanh\phi + \mathrm{i}\lambda] \exp(-\mathrm{i}\lambda\phi/\eta) + A_{-1}(t)[\eta\tanh\phi - \mathrm{i}\lambda]\exp(\mathrm{i}\lambda\phi/\eta). \tag{A6}$$

We must also require that this solution satisfy the time evolution given by (A2). This along with the boundary conditions (A3) and (A4) will determine $A_1(t)$ and $A_{-1}(t)$.

We let $v = v_1 + v_{-1}$, where

$$v_\alpha = C_\alpha[\eta\tanh\phi + \mathrm{i}\alpha\lambda] \qquad C_\alpha \equiv A_\alpha e^{-i\alpha\lambda\phi/\eta} \qquad \alpha = \pm 1. \tag{A7}$$

Then we have

$$\begin{aligned} C_{\alpha x} &= -\mathrm{i}\alpha\lambda C_\alpha \\ v_{\alpha x} &= C_{\alpha t}[\eta\tanh\phi + \mathrm{i}\alpha\lambda] + 4\eta^4 C_\alpha \operatorname{sech}^2\phi \\ v_{\alpha x} &= C_{\alpha x}[\eta\tanh\phi + \mathrm{i}\alpha\lambda] + \eta^2 C_\alpha \operatorname{sech}^2\phi. \end{aligned} \tag{A8}$$

Finally, if we substitute (A1) into (A2) for v_{xxx}, we obtain the equation

$$v_t = (4\lambda^2 - 2q)v_x + q_x v + \gamma v. \tag{A9}$$

Inserting the expressions (A8) into (A9), we find the time dependence of C_α:

$$C_{\alpha t} = [-4\mathrm{i}\alpha\lambda^3 + \gamma]C_\alpha \tag{A10}$$

which gives the general solution for v,

$$\begin{aligned} v(x,t) = A[\eta\tanh\phi + \mathrm{i}\lambda]\exp[-\mathrm{i}\lambda(x - x_0) + (\gamma - 4\mathrm{i}\lambda^3)t] \\ + B[\eta\tanh\phi - \mathrm{i}\lambda]\exp[\mathrm{i}\lambda(x - x_0) + (\gamma + 4\mathrm{i}\lambda^3)t]. \end{aligned} \tag{A11}$$

For a fixed time, we look at ϕ_2 as $x \to -\infty$ and ψ_2 as $x \to \infty$ in (A11) to find

$$\gamma = \begin{cases} 4\mathrm{i}\lambda^3 & \text{for } \phi_2 \quad \text{(A12)} \\ -4\mathrm{i}\lambda^3 & \text{for } \psi_2. \quad \text{(A13)} \end{cases}$$

Applying the boundary conditions (A3) and (A4) to the general solution yields ϕ_2, ψ_2:

$$\phi_2(\phi, t; \lambda) = \frac{\exp(-\mathrm{i}\lambda\phi/\eta - 4\mathrm{i}\lambda^2 t)}{(\mathrm{i}\lambda - \eta)}[\eta\tanh\phi + \mathrm{i}\lambda] \tag{A14}$$

$$\psi_2(\phi, t; \lambda) = \frac{\exp(\mathrm{i}\lambda\phi/\eta + 4\mathrm{i}\lambda^2 t)}{(\eta - \mathrm{i}\lambda)}[\eta\tanh\phi - \mathrm{i}\lambda]. \tag{A15}$$

The eigenfunctions, which we have discussed in the text, and their adjoints may now be calculated. They are

$$\Phi(x,t;\lambda) = \frac{\exp(-2\mathrm{i}\lambda\phi/\eta - 8\mathrm{i}\lambda\eta^2 t)}{(\mathrm{i}\lambda-\eta)^2}\,[\eta^2\tanh^2\phi + 2\mathrm{i}\lambda\eta\tanh\phi - \lambda^2] \tag{A16}$$

$$\Phi_1(x,t) = \tfrac{1}{4}\exp(8\eta^3 t)\,\mathrm{sech}^2\phi \tag{A17}$$

$$\chi_1(x,t) = \frac{\mathrm{i}}{2\eta}\exp(8\eta^3 t)[(\tfrac{1}{2}-\phi-4\eta^3 t)\,\mathrm{sech}^2\phi - \tanh\phi - 1] \tag{A18}$$

$$\Phi^{\mathrm{A}}(x,t;\lambda) = 2\frac{\exp(2\mathrm{i}\lambda\phi/\eta + 8\mathrm{i}\lambda\eta^2 t)}{(\mathrm{i}\lambda-\eta)^2}[-\eta^3\tanh^3\phi + 2\mathrm{i}\lambda\eta^2\tanh^2\phi + (2\lambda^2\eta+\eta^3)\tanh\phi - \mathrm{i}(\lambda^3+\lambda\eta^2)] \tag{A19}$$

$$\Phi_1^{\mathrm{A}} = -\frac{\eta}{2}\exp(-8\eta^3 t)\,\mathrm{sech}^2\phi\tanh\phi \tag{A20}$$

$$\chi_1^{\mathrm{A}} = \mathrm{i}\exp(-8\eta^3 t)[\mathrm{sech}^2\phi - (\tfrac{1}{2}+\phi+4\eta^3 t)\,\mathrm{sech}^2\phi\tanh\phi] \tag{A21}$$

$$\Psi(x,t;\lambda) = \frac{\exp(2\mathrm{i}\lambda\phi/\eta + 8\mathrm{i}\lambda\eta^2 t)}{(\mathrm{i}\lambda-\eta)^2}[\eta^2\tanh^2\phi - 2\mathrm{i}\lambda\eta\tanh\phi - \lambda^2] \tag{A22}$$

$$\Psi_1(x,t) = \tfrac{1}{4}\exp(-8\eta^3 t)\,\mathrm{sech}^2\phi \tag{A23}$$

$$\tau_1(x,t) = \frac{\mathrm{i}}{2\eta}\exp(-8\eta^3 t)[(\tfrac{1}{2}+\phi+4\eta^3 t)\,\mathrm{sech}^2\phi + \tanh\phi + 1] \tag{A24}$$

$$\Psi^{\mathrm{A}}(x,t;\lambda) = \frac{2\exp(-2\mathrm{i}\lambda\phi/\eta - 8\mathrm{i}\lambda\eta^2 t)}{(\mathrm{i}\lambda-\eta)^2}[-\eta^3\tanh^3\phi - 2\mathrm{i}\lambda\eta^2\tanh^2\phi + (2\lambda^2\eta+\eta^3)\tanh\phi + \mathrm{i}(\lambda^3+\lambda\eta^2)] \tag{A25}$$

$$\Psi_1^{\mathrm{A}} = -\tfrac{1}{2}\eta\exp(8\eta^3 t)\,\mathrm{sech}^2\phi\tanh\phi \tag{A26}$$

$$\tau_1^{\mathrm{A}} = -\mathrm{i}\exp(8\eta^3 t)[\mathrm{sech}^2\phi + (\tfrac{1}{2}-\phi-4\eta^3 t)\,\mathrm{sech}^2\phi\tanh\phi] \tag{A27}$$

$$\Lambda_1(x,t) = -\frac{\mathrm{i}}{\eta}\exp(8\eta^3 t)[(\phi+4\eta^3 t)\,\mathrm{sech}^2\phi + \tanh\phi] \tag{A28}$$

$$\Lambda_1^{\mathrm{A}} = 2\mathrm{i}\exp(-8\eta^3 t)[\mathrm{sech}^2\phi - (\phi+4\eta^3 t)\,\mathrm{sech}^2\phi\tanh\phi]. \tag{A29}$$

The orthogonality relations for these states can be found through the use of the Zakharov–Shabat eigenvalue problem [40]

$$v_{1x} + \mathrm{i}\lambda v_1 = q v_2 \qquad v_{2x} - \mathrm{i}\lambda v_2 = -v_1 \tag{A30}$$

$$v_{2xx} + (\lambda^2 + q)v_2 = 0. \tag{A31}$$

The basis functions are of the form v_{2x}^2, while the adjoint functions are of the form w_2^2. The various inner products, which we need, are of the general form

$$\int_{-\infty}^{\infty} v_{2x}^2 w_2^2\,\mathrm{d}x = \langle v_2^2 | w_2^2\rangle. \tag{A32}$$

From (A30) and (A31) we have

$$4\lambda^2 v_2^2 = 2(v_1^2 - 2\mathrm{i}\lambda v_1 v_2) - (v_2^2)_{xx} - 2qv_2^2 \tag{A33}$$

$$(v_1^2 - 2\mathrm{i}\lambda v_1 v_2)_x = -q(v_2^2)_x. \tag{A34}$$

Writing similar equations for $w(\lambda')$, we have

$$\begin{aligned}4(\lambda^2 - \lambda'^2)v_{2x}^2 w_2^2 = {} & w_2^2\{2(v_1^2 - 2\mathrm{i}\lambda v_1 v_2)_x - (v_2^2)_{xxx} - 2(qv_2^2)_x\} \\ & - v_{2x}^2\{2(w_1^2 - 2\mathrm{i}\lambda' w_1 w_2) - (w_2^2)_{xx} - 2qw_2^2\}.\end{aligned} \tag{A35}$$

Using (A34) we find that the right-hand side of this equation is a divergence [40]:

$$4(\lambda^2 - \lambda'^2)v_{2x}^2 w_2^2 = \partial_x\{w_{2x}^2 v_{2x}^2 - w_2^2 v_{2xx}^2 - 2v_2^2(w_1^2 - 2\mathrm{i}\lambda' w_1 w_2) - 2v_2^2 q w_2^2\}. \tag{A36}$$

Integrating from $-R$ to R, we obtain the general relation

$$\int_{-R}^{R} \mathrm{d}x\, v_{2x}^2 w_2^2 = \frac{1}{4(\lambda^2 - \lambda'^2)}[w_{2x}^2 v_{2x}^2 - w_2^2 v_{2xx}^2 - 2v_2^2(w_1^2 - 2\mathrm{i}\lambda' w_1 w_2)]_{-R}^{R}. \tag{A37}$$

Using the Jost functions for the Zakharov–Shabat eigenvalue problem with the boundary conditions

$$\begin{aligned}&\phi \sim \begin{pmatrix} 2\mathrm{i}\lambda \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}\lambda x} \quad x \to -\infty \qquad \phi \sim \begin{pmatrix} 2\mathrm{i}\lambda a\, \mathrm{e}^{-\mathrm{i}\lambda x} \\ a\,\mathrm{e}^{-\mathrm{i}\lambda x} + b\,\mathrm{e}^{\mathrm{i}\lambda x} \end{pmatrix} \quad x \to +\infty \\ &\psi \sim \begin{pmatrix} -2\mathrm{i}\lambda \hat{b}\, \mathrm{e}^{-\mathrm{i}\lambda x} \\ a\,\mathrm{e}^{\mathrm{i}\lambda x} - \hat{b}\,\mathrm{e}^{-\mathrm{i}\lambda x} \end{pmatrix} \quad x \to -\infty \qquad \psi \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}\lambda x} \quad x \to +\infty\end{aligned} \tag{A38}$$

we can compute the required inner products. In particular we find for the states derived above, the non-zero products are:

$$\langle \Phi^{\mathrm{A}}(\lambda')|\Phi(\lambda)\rangle = 2\pi\mathrm{i}\lambda a^2(\lambda)\delta(\lambda - \lambda') \tag{A39}$$

$$\langle \Psi^{\mathrm{A}}(\lambda')|\Psi(\lambda)\rangle = -2\pi\mathrm{i}\lambda a^2(\lambda)\delta(\lambda - \lambda') \tag{A40}$$

$$\begin{aligned}\langle \Phi^{\mathrm{A}}(\lambda')|\Psi(\lambda)\rangle &= -\langle \Psi^{\mathrm{A}}(\lambda')|\Phi(\lambda)\rangle \\ &= 2\pi\mathrm{i}\lambda[a^2\hat{a}^2 - b^2\hat{b}^2]\delta(\lambda + \lambda') + 4\lambda'^2\hat{a}(\lambda')a(\lambda)b(\lambda)\hat{b}(\lambda')\end{aligned} \tag{A41}$$

$$\langle \Phi_k^{\mathrm{A}}|\chi_l\rangle = \langle \chi_k^{\mathrm{A}}|\Phi_l\rangle = -\lambda_k {a'_k}^2 \delta_{lk} = \tfrac{1}{2}\langle \Lambda_k^{\mathrm{A}}|\Phi_l\rangle = \tfrac{1}{2}\langle \Phi_l^{\mathrm{A}}|\Lambda_k\rangle \tag{A42}$$

$$\langle \Psi_k^{\mathrm{A}}|\tau_l\rangle = \langle \tau_k^{\mathrm{A}}|\Psi_l\rangle = \lambda_k {a'_k}^2 \delta_{kl} \tag{A43}$$

$$\langle \chi_k^{\mathrm{A}}|\chi_l\rangle = -\langle \tau_k^{\mathrm{A}}|\tau_l\rangle = -a'_k(\lambda_k a''_k + a'_k)\delta_{kl}. \tag{A44}$$

Appendix B. Asymptotics for the damped KdV solution

The first-order solution for the damped KdV was given in equation (73). We now compare this solution with those given by Kaup and Newell [30] and Knickerbocker and Newell [33], and carry out an asymptotic analysis for large times.

Recalling the first-order solution

$$u_1 = \int_{-\infty}^{\infty} f(\lambda, t)\Phi^{\mathrm{A}}(\phi, t; \lambda)\, \mathrm{d}\lambda \tag{B1}$$

where

$$f(\lambda, t) = \frac{\gamma[\exp(-8\mathrm{i}\lambda\eta^2 t) - \exp(8\mathrm{i}\lambda^3 t)]}{12\lambda a^2(\lambda)(\eta^2 + \lambda^2)\sinh(\pi\lambda/\eta)} \tag{B2}$$

and from appendix A we write

$$\begin{aligned} \Phi^{\mathrm{A}}(\phi, t; \lambda) &= \frac{\partial}{\partial x}\left[\exp[2\mathrm{i}\lambda(x - x_0)]\left(1 - \frac{2\eta}{\eta - \mathrm{i}\lambda}\frac{\mathrm{e}^{2\phi}}{1 + \mathrm{e}^{2\phi}}\right)^2\right] \\ x_0 &\equiv \bar{x} - 4\eta^2 t. \end{aligned} \tag{B3}$$

Combining these equations, and partially carrying out the differentiation, we have

$$\begin{aligned} u_1 = \frac{\gamma}{12}\int_{-\infty}^{\infty} & \frac{[\exp(-8\mathrm{i}\lambda\eta^2 t) - \exp(8\mathrm{i}\lambda^3 t)]\exp[2\mathrm{i}\lambda(x - x_0)]}{\lambda a^2(\lambda)(\eta^2 + \lambda^2)\sinh(\pi\lambda/\eta)} \\ & \times (2\mathrm{i}\lambda + \eta\partial_\phi)\left(1 - \frac{2\eta}{\eta - \mathrm{i}\lambda}\frac{\mathrm{e}^{2\phi}}{1 + \mathrm{e}^{2\phi}}\right)^2. \end{aligned} \tag{B4}$$

This can be written as the sum of the two terms

$$\begin{aligned} u_{1a} &= -\frac{4\gamma}{3}\int_{-\infty}^{\infty} \frac{\lambda\, \mathrm{d}\lambda}{a(\lambda)\sinh(\pi\lambda/\eta)}\left(1 - \frac{2\eta}{\eta - \mathrm{i}\lambda}\frac{\mathrm{e}^{2\phi}}{1 + \mathrm{e}^{2\phi}}\right)^2 \exp[2\mathrm{i}\lambda(x - x_0)] \\ &\qquad \times \left(\frac{\exp(-8\mathrm{i}\lambda\eta^2 t) - \exp(8\mathrm{i}\lambda^3 t)}{8\mathrm{i}\lambda(\lambda^2 + \eta^2)}\right) \end{aligned} \tag{B5}$$

$$\begin{aligned} u_{1b} = \frac{\eta\gamma}{12}\int_{-\infty}^{\infty} & \frac{[\exp(-8\mathrm{i}\lambda\eta^2 t) - \exp(8\mathrm{i}\lambda^3 t)]\exp[2\mathrm{i}\lambda(x - x_0)]}{\lambda a^2(\lambda)(\eta^2 + \lambda^2)\sinh(\pi\lambda/\eta)} \\ & \partial_\phi\left(1 - \frac{2\eta}{\eta - \mathrm{i}\lambda}\frac{\mathrm{e}^{2\phi}}{1 + \mathrm{e}^{2\phi}}\right)^2. \end{aligned} \tag{B6}$$

The first term gives the integral obtained originally by Kaup and Newell [30], and later compared to numerical simulations by Knickerbocker and Newell [33].

We begin the analysis of equation (B4) by carrying out the differentiation and writing it as

$$\begin{aligned} u_1 &= \frac{\gamma}{6}\int_{-\infty}^{\infty} \frac{[\exp(-8\mathrm{i}\lambda\eta^2 t) - \exp(8\mathrm{i}\lambda^3 t)]\exp[2\mathrm{i}\lambda(x - x_0)]}{\lambda(\eta^2 + \lambda^2)\sinh(\pi\lambda/\eta)} h(\lambda) \\ h(\lambda) &= 2\mathrm{i}\lambda\eta^2\tanh^2\phi - \mathrm{i}\lambda(\eta^2 + \lambda^2) + 2\eta\lambda^2\tanh\phi + \eta^3\,\mathrm{sech}^2\phi\tanh\phi. \end{aligned} \tag{B7}$$

Following Kaup and Newell, we seek the solution for large x, t, while keeping γt fixed. The leading contribution may be found by invoking the Riemann–Lebesgue lemma [43], which tells us that if f is $L_1(-\infty,\infty)$, then

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos nx\,\mathrm{d}x = 0.$$

Thus, the major contribution to the integrals in (B5) and (B6) comes from values of λ near $\lambda = 0$. Taking the limit $\lambda \to 0$, we have in the region of ϕ such that $\operatorname{sech}^2\phi \approx 0$

$$u_1 \sim \frac{\gamma}{6\pi\eta}(2\tanh^2\phi - 1)\int_{-\infty}^{\infty}\frac{\sin 2\lambda(x-x_0-4\eta^2 t) - \sin 2\lambda(x-x_0+4\eta^2 t)}{2\lambda}\,\mathrm{d}(2\lambda). \tag{B8}$$

We note that this is the almost same answer as Kaup and Newell had obtained; the difference is the factor $2\tanh^2\phi - 1$ instead of their $\tanh^2\phi$ [30].

The first term in (B8) can be integrated using [8]

$$\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{\sin mx}{x} = \frac{\pi}{2}\operatorname{sgn}(m) \tag{B9}$$

to give

$$\int_{-\infty}^{\infty}\mathrm{d}(2\lambda)\,\frac{\sin 2\lambda(x-x_0-4\eta^2 t)}{2\lambda} = \pi\operatorname{sgn}(x-x_0-4\eta^2 t). \tag{B10}$$

The second term can be integrated by relating it to the Airy function [5]

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(\tfrac{1}{3}\tau^3 + z\tau)\,\mathrm{d}\tau \tag{B11}$$

and making the transformation

$$\begin{aligned}\tau &= (3t)^{1/3}\lambda\\ z &= (3t)^{-1/3}(x-x_0).\end{aligned} \tag{B12}$$

Integrating equation (B11), we have

$$\begin{aligned}2\pi\int_0^{z_0}&\mathrm{Ai}(z)\,\mathrm{d}z\\ &= \int_{-\infty}^{\infty}\mathrm{d}\tau\int_0^{z_0}\mathrm{d}z\cos(\tfrac{1}{3}\tau^3 + z\tau)\\ &= \int_{-\infty}^{\infty}\mathrm{d}(2\lambda)\,\frac{\sin 2\lambda(x-x_0-4\eta^2 t)}{2\lambda} - \tfrac{1}{3}\pi\operatorname{sgn}(x-x_0).\end{aligned} \tag{B13}$$

Therefore, we have for the first-order solution

$$u_1 \sim \frac{\gamma}{6\pi\eta}(2\tanh^2\phi - 1)\left(\pi\operatorname{sgn}(x-x_0-4\eta^2 t) - \tfrac{1}{3}\pi\operatorname{sgn}(x-x_0) - 2\pi\int_0^{z_0}\mathrm{Ai}(z)\,\mathrm{d}z\right). \tag{B14}$$

The term in the brackets was also obtained by Knickerbocker and Newell [33].

We now make use of the asymptotics of the Airy function

$$\mathrm{Ai}(s) \sim \frac{s^{-1/4}}{2\sqrt{\pi}} \exp[-\tfrac{2}{3}s^{3/2}] \qquad s \to \infty \tag{B15}$$

$$\mathrm{Ai}(s) \sim \frac{|s|^{-1/4}}{\sqrt{\pi}} \sin[\tfrac{2}{3}|s|^{3/2} + \tfrac{1}{4}\pi] \qquad s \to -\infty \tag{B16}$$

and the Mellin transform [5]

$$M[f(x), t] = \int_0^\infty y^{t-1} f(y)\,\mathrm{d}y \tag{B17}$$

to complete this analysis. We first consider the integral

$$I = \int_0^{z_0} \mathrm{Ai}(z)\,\mathrm{d}z \tag{B18}$$

for z_0 positive. It can be written as

$$I = \int_0^\infty \mathrm{Ai}(z)\,\mathrm{d}z - \int_{z_0}^\infty \mathrm{Ai}(z)\,\mathrm{d}z = M[\mathrm{Ai}(x), 1] - \int_{z_0}^\infty \mathrm{Ai}(z)\,\mathrm{d}z \tag{B19}$$

where $M[f(x), t]$ is the Mellin transform. For the Airy function we have [5]

$$M[\mathrm{Ai}(z), t] = \frac{3^{2t/3-7/6}}{2\pi} \Gamma(\tfrac{1}{3}t)\Gamma(\tfrac{1}{3}(t+1)) \tag{B20}$$

giving $M[\mathrm{Ai}(x), 1] = 1/3$.

We can obtain the large behaviour of the second integral in equation (B19) using the asymptotic expression for the Airy function [5]

$$\mathrm{Ai}(z) \sim \frac{z^{-1/4}}{2\sqrt{\pi}} \exp[-\tfrac{2}{3}z^{3/2}] \qquad s \to \infty \tag{B21}$$

since for large z_0 the range of the integration variable is always large. Inserting (B21) in the second integral, and integrating by parts, we have

$$\begin{aligned}\int_{z_0}^\infty \mathrm{Ai}(z)\,\mathrm{d}z &\sim -\frac{1}{2\sqrt{\pi}} {z_0}^{-3/4} \exp(-\tfrac{2}{3}{z_0}^{3/2}) + \frac{1}{2\sqrt{\pi}} \int_{z_0}^\infty z^{-5/4} \frac{\mathrm{d}}{\mathrm{d}z}(\exp(-\tfrac{2}{3}z^{3/2}))\,\mathrm{d}z \\ &= -\frac{1}{2\sqrt{\pi}} {z_0}^{-3/4} \exp(-\tfrac{2}{3}{z_0}^{3/2}) + \mathrm{O}({z_0}^{-3/4} \exp(-\tfrac{2}{3}{z_0}^{3/2})).\end{aligned} \tag{B22}$$

For $z_0 < 0$, the analysis is similar. Rewriting the integral, we have

$$\begin{aligned} I = -\int_0^{|z_0|} \mathrm{Ai}(-z)\,\mathrm{d}z &= -\int_0^\infty \mathrm{Ai}(-z)\,\mathrm{d}z + \int_{|z_0|}^\infty \mathrm{Ai}(-z)\,\mathrm{d}z \\ &= -M[\mathrm{Ai}(-z), 1] + \int_{|z_0|}^\infty \mathrm{Ai}(-z)\,\mathrm{d}z.\end{aligned} \tag{B23}$$

We make use of the expressions (B16) and

$$M[\mathrm{Ai}(-z),t] = \frac{3^{2t/3-7/6}}{\pi}\Gamma(\tfrac{1}{3}t)\Gamma(\tfrac{1}{3}(t+1))\sin(\tfrac{1}{3}\pi t+\tfrac{1}{6}\pi) \tag{B24}$$

$$M[\mathrm{Ai}(-z),1] = \tfrac{1}{3} \tag{B25}$$

to obtain

$$\int_{|z_0|}^{\infty} \mathrm{d}z\ \mathrm{Ai}(-z) \sim \frac{1}{\sqrt{\pi}}|z_0|^{-3/4}\cos[\tfrac{2}{3}|z_0|^{3/2}+\tfrac{1}{4}\pi] + \mathrm{O}(|z_0|^{-5/4}). \tag{B26}$$

Putting these two cases together, we have the required expression for the equation

$$2\pi\int_0^{z_0}\mathrm{Ai}(z)\,\mathrm{d}z \sim \frac{2\pi}{3}\operatorname{sgn}(z_0) + \begin{cases} -\sqrt{\pi}\,z_0{}^{-3/4}\exp(-\tfrac{2}{3}z_0{}^{3/2}) + \mathrm{O}(z_0{}^{-3/4}\exp(-\tfrac{2}{3}z_0{}^{3/2})) & z_0>0 \\ 2\sqrt{\pi}\,|z_0|^{-3/4}\cos[\tfrac{2}{3}|z_0|^{3/2}+\tfrac{1}{4}\pi] + \mathrm{O}(|z_0|^{-5/4}) & z_0<0. \end{cases} \tag{B27}$$

Inserting this into (B14) we arrive at the asymptotic form for the first-order solution u_1 :

$$\begin{cases} \dfrac{\gamma}{6\sqrt{\pi}\,\eta_0}\left(\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right)^{-3/4}\exp\left[-\dfrac{2}{3}\left(\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right)^{3/2}\right] & \text{for } 0<4\eta_0^2t<\xi \\ -\dfrac{\gamma}{3\eta_0}+\dfrac{\gamma}{6\sqrt{\pi}\,\eta_0}\left(\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right)^{-3/4}\exp\left[-\dfrac{2}{3}\left(\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right)^{3/2}\right] & \text{for } 0<\xi<4\eta_0^2t \\ -\dfrac{\gamma}{3\sqrt{\pi}\eta_0}\left|\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right|^{-3/4}\cos\left[\dfrac{2}{3}\left|\dfrac{x-\bar{x}_0}{(3t)^{1/3}}\right|^{3/2}+\dfrac{\pi}{4}\right] & \text{for } \xi<0 \end{cases} \tag{B28}$$

for large times, and in the region where $\tanh^2\phi \simeq 1$. Here we have defined

$$\xi \equiv x - \bar{x}_0 \qquad \bar{x}_0 \equiv \bar{x} - 4\eta_0^2 t. \tag{B29}$$

Appendix C. Orthogonality relations for the sine-Gordon bases

In section 5 we have shown that the proper states in which to expand the first order solution of the perturbed sine-Gordon equation in lightcone coordinates were of the form $\phi_1^2+\phi_2^2$ or $\psi_1^2+\psi_2^2$. That such an expansion is allowed depends on the closure of these states. Also, certain orthogonality relations were needed, equations (166), in order to compute the expansion coefficients. In this appendix, we shall review some of Kaup's results on the closure of the squared eigenfunctions of the Zakharov–Shabat eigenvalue problem [27]. The assumed information, which we needed for the sine-Gordon problem, will then be derived from these results.

The starting place is the Zakharov–Shabat eigenvalue problem (152), and the Jost solutions, which are defined through the boundary conditions (6)–(9). Employing the bra–ket notation of Dirac, Kaup obtains the complete set of states [27]

$$\Psi(\lambda, x) = \begin{bmatrix} \psi_1^2 \\ \psi_2^2 \end{bmatrix} = \langle x|1, \lambda\rangle \tag{C1}$$

$$\bar{\Psi}(\lambda, x) = \begin{bmatrix} \bar{\psi}_1^2 \\ \bar{\psi}_2^2 \end{bmatrix} = \langle x|2, \lambda\rangle \tag{C2}$$

$$\langle x|1, \lambda_k\rangle = \langle x|1, \lambda\rangle_{\lambda=\lambda_k} \qquad \langle x|2, \bar{\lambda}_k\rangle$$

$$\langle x|1P, \lambda_k\rangle = \frac{\partial}{\partial\lambda}\langle x|1, \lambda\rangle_{\lambda=\lambda_k} \qquad \langle x|2P, \lambda_k\rangle \tag{C3}$$

and their adjoints

$$[\Psi^{\mathrm{A}}(\lambda, x)]^{\mathrm{T}} = \langle 1, \lambda|x\rangle = (\phi_2^2, -\phi_1^2) \tag{C4}$$

$$[\bar{\Psi}^{\mathrm{A}}(\lambda, x)]^{\mathrm{T}} = \langle 2, \lambda|x\rangle = (\bar{\phi}_2^2, -\bar{\phi}_1^2) \tag{C5}$$

$$\langle 1, \lambda_k|x\rangle \qquad \langle 2, \bar{\lambda}_k|x\rangle$$

$$\langle 1P, \lambda_k|x\rangle \qquad \langle 2P, \bar{\lambda}_k|x\rangle \tag{C6}$$

The non-zero inner products between these states are

$$\langle 1, \lambda'|1, \lambda\rangle = -\pi a^2(\lambda)\delta(\lambda - \lambda') \tag{C7}$$

$$\langle 1, \lambda'|2, \lambda\rangle = \langle 2, \lambda'|1, \lambda\rangle = 0 \tag{C8}$$

$$\langle 2, \lambda'|2, \lambda\rangle = \pi\bar{a}^2(\lambda)\delta(\lambda - \lambda') \tag{C9}$$

$$\langle 1P, \lambda_k|1P, \lambda_l\rangle = -\tfrac{1}{2}a_k''a_k'\delta_l^k \tag{C10}$$

$$\langle 1P, \lambda_k|1, \lambda_l\rangle = \langle 1, \lambda_l|1P, \lambda_k\rangle = -\tfrac{1}{2}(a_k')^2\delta_l^k \tag{C11}$$

$$\langle 2P, \bar{\lambda}_k|2, \bar{\lambda}_l\rangle = \langle 2, \bar{\lambda}_l|2P, \bar{\lambda}_k\rangle = -\tfrac{1}{2}(\bar{a}_k')^2\delta_l^k \tag{C12}$$

$$\langle 2P, \bar{\lambda}_k|2P, \bar{\lambda}_l\rangle = -\tfrac{1}{2}\bar{a}_k''\bar{a}_k'\delta_l^k \tag{C13}$$

where

$$a_k' \equiv \frac{\partial}{\partial\lambda}\, a(\lambda)_{\lambda=\lambda_k} \qquad \bar{a}_k' \equiv \frac{\partial}{\partial\bar{\lambda}}\, a(\lambda)_{\lambda=\bar{\lambda}_k} \tag{C14}$$

$$a_k'' = \frac{\partial^2}{\partial\lambda^2}\, a(\lambda)_{\lambda=\lambda_k} \qquad \bar{a}_k'' = \frac{\partial^2}{\partial\bar{\lambda}^2}\, \bar{a}(\lambda)_{\lambda=\bar{\lambda}_k}. \tag{C15}$$

δ_l^k is the Kronecker delta, and $\delta(\lambda - \lambda')$ is the Dirac delta function. For the one-soliton case $a(\lambda)$ is given by (30). Finally, Kaup expresses an expansion of a two-component function

$$\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = f(x) = \langle x|f\rangle$$

by

$$|f\rangle = \int_{-\infty}^{\infty} \frac{d\lambda}{\pi} [f(\lambda)|1,\lambda\rangle + \bar{f}(\lambda)|2,\lambda\rangle] + \sum_{k=1}^{N}[f_k|1,\lambda_k\rangle + g_k|1P,\lambda_k\rangle] + \sum_{k=1}^{\bar{N}}[\bar{f}_k|1,\bar{\lambda}_k\rangle + \bar{g}_k|1P,\bar{\lambda}_k\rangle]. \tag{C16}$$

Now, we want to derive the orthogonality relations in (166). Recall that in one basis, we have $\hat{\Omega} = \psi_2^2 - \psi_1^2$, while in the second basis, $\hat{\Omega} = \phi_2^2 - \phi_1^2$. Our aim is to integrate the product $\hat{\Omega}^A\hat{\Omega}$, where $\hat{\Omega}^A = \phi_1^2 + \phi_2^2$, $\psi_1^2 + \psi_2^2$, respectively. Noting that for the sine-Gordon,

$$\begin{aligned} \bar{\Psi}(\lambda, x) &= \begin{bmatrix} \psi_2^2(-\lambda) \\ \psi_1^2(-\lambda) \end{bmatrix} \\ \bar{\Psi}^A(\lambda, x)^T &= (\phi_1^2(-\lambda), -\phi_2^2(\lambda)) \end{aligned} \tag{C17}$$

we can get this product using the first basis, by

$$\begin{aligned} 2\hat{\Omega}\hat{\Omega}^A &= \begin{pmatrix} \hat{\Omega}(\lambda') \\ -\hat{\Omega}(\lambda') \end{pmatrix}^T \begin{pmatrix} \hat{\Omega}^A(\lambda) \\ -\hat{\Omega}^A(\lambda) \end{pmatrix} \\ &= [\bar{\Psi}^A(-\lambda') + \Psi(\lambda')][\bar{\Psi}(-\lambda) - \Psi(\lambda)] \end{aligned} \tag{C18}$$

and through the second basis by

$$2\hat{\Omega}\hat{\Omega}^A = \begin{pmatrix} \hat{\Omega}(\lambda') \\ \hat{\Omega}(\lambda') \end{pmatrix}^T \begin{pmatrix} \hat{\Omega}^A(\lambda) \\ \hat{\Omega}^A(\lambda) \end{pmatrix} = [\Psi^A(\lambda') - \bar{\Psi}(-\lambda')][\Psi(\lambda) + \bar{\Psi}(-\lambda)]. \tag{C19}$$

Expressing this in the Dirac notation, and combining (C18) and (C19), we have the general product

$$\begin{aligned} 2\hat{\Omega}\hat{\Omega}^A = \pm\{&-\langle 1,\lambda'|x\rangle\langle x|1,\lambda\rangle \pm \langle 1,\lambda'|x\rangle\langle x|2,-\lambda\rangle \\ &\mp \langle 2,-\lambda'|x\rangle\langle x|1,\lambda\rangle + \langle 2,-\lambda'|x\rangle\langle x|2,-\lambda\rangle\} \end{aligned} \tag{C20}$$

where the upper sign refers to the first basis

$$\hat{\Omega} = (\psi_2^2 - \psi_1^2)(\lambda) \qquad \hat{\Omega}^A = (\phi_1^2 + \phi_2^2)(\lambda) \tag{C21}$$

and the lower sign refers to the second basis

$$\hat{\Omega} = (\phi_2^2 - \phi_1^2)(\lambda) \qquad \hat{\Omega}^A = (\psi_1^2 + \psi_2^2)(\lambda) \tag{C22}$$

Integrating (C20) over x, and using $a(-\lambda) = \bar{a}(\lambda)$ and equations (C7)–(C9), gives

$$\langle \hat{\Omega}^A(\lambda')|\hat{\Omega}(\lambda)\rangle = \pm\pi a^2(\lambda)\delta(\lambda - \lambda'). \tag{C23}$$

Similarly, we find

$$\langle \hat{\Omega}^A(\lambda)|\hat{\Omega}_1\rangle = \langle \hat{\Omega}_1^A|\hat{\Omega}(\lambda)\rangle = 0. \tag{C24}$$

In order to find the products with $\hat{\Lambda}_1 = \partial_\lambda \hat{\Omega}_{\lambda_1}$, or $\hat{\Lambda}_1^A = \partial_\lambda \hat{\Omega}_{\lambda_1}^A$, we note that in the first basis

$$\hat{\Lambda}_1 = -\langle x|2P, -\lambda_1\rangle - \langle x|1P, \lambda_1\rangle \tag{C25}$$

and

$$\hat{\Lambda}_1^A = -\langle 2P, \lambda_1|x\rangle + \langle 1P, \lambda|x\rangle \tag{C26}$$

and for the second basis

$$\hat{\Lambda}_1 = \langle 1P, \lambda_1|x\rangle + \langle 2P, \lambda|x\rangle \tag{C27}$$

and

$$\hat{\Lambda}_1^A = \langle x|1P, \lambda_1\rangle - \langle x|2P, \lambda_1\rangle. \tag{C28}$$

Using this and the general product (C20), we find

$$\begin{aligned} 2\hat{\Lambda}_1\hat{\Omega}_1 &= 2\partial_{\lambda'}[\hat{\Omega}(\lambda')\hat{\Omega}(\lambda)]_{\lambda'=\lambda_1,\lambda=\lambda_1} \\ &= \pm\{-\langle 1P, \lambda_1|x\rangle\langle x|1P, \lambda_1\rangle \pm \langle 1P, \lambda_1|x\rangle\langle x|2, -\lambda_1\rangle \\ &\quad \mp \langle 2P, -\lambda_1|x\rangle\langle x|1P, \lambda_1\rangle + \langle 2P, -\lambda_1|x\rangle\langle x|2P, -\lambda_1\rangle\} \end{aligned} \tag{C29}$$

which integrates to:

$$\langle \hat{\Lambda}_1^A|\hat{\Omega}_1\rangle = \pm\tfrac{1}{2}[\tfrac{1}{2}\mathrm{i}(a_1')^2 + \tfrac{1}{2}\mathrm{i}(a_1')^2] = \mp\frac{\mathrm{i}}{4\eta^2}. \tag{C30}$$

Similarly, we have

$$\langle \hat{\Lambda}_1|\hat{\Omega}_1^A\rangle = \mp\frac{1}{4\eta^2} \qquad \langle \hat{\Lambda}_1^A|\hat{\Lambda}_1\rangle = 0. \tag{C31}$$

All other inner products vanish. Thus, we have derived the stated orthogonality relations (166) from the known results of Kaup. We now turn to the expansion (C16).

Using the orthogonality relations (C7)-(C13), Kaup computes the expansion coefficients as

$$f(\lambda) = -\frac{1}{a^2(\lambda)}\langle 1, \lambda|f\rangle \tag{C32}$$

$$\bar{f}(\lambda) = \frac{1}{\bar{a}^2(\lambda)}\langle 2, \lambda|f\rangle \tag{C33}$$

$$g_k = \frac{2\mathrm{i}}{{a_k'}^2}\langle 1, \lambda_k|f\rangle \tag{C34}$$

$$f_k = \frac{2\mathrm{i}}{{a_k'}^2}\langle 1P, \lambda_k|f\rangle - \frac{2\mathrm{i}}{(a_k')^3}a_k''\langle 1, \lambda_k|f\rangle \tag{C35}$$

$$\bar{g}_k = \frac{2\mathrm{i}}{(\bar{a}_k')^2}\langle 2, \bar{\lambda}_k|f\rangle \tag{C36}$$

$$\bar{f}_k = \frac{2\mathrm{i}}{(\bar{a}_k')^2}\langle 1P, \bar{\lambda}_k|f\rangle - \frac{2\mathrm{i}}{(\bar{a}_k')^3}\bar{a}_k''\langle 1, \bar{\lambda}_k|f\rangle. \tag{C37}$$

In our problem, we take $N = \bar{N} = 1$.

In order to simplify the discussion, we will only consider the sum over the continuous spectrum. The analysis for the full expansion proceeds in a similar fashion. We start with the expansion in a *coordinate representation* as

$$f(x) = \langle x|f\rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{\pi}\,[f(\lambda)\langle x|1,\lambda\rangle + \bar{f}(\lambda)\langle x|2,\lambda\rangle]. \tag{C38}$$

Noting that

$$\begin{aligned}\langle x|1,\lambda\rangle &= \begin{pmatrix}\psi_1^2\\ \psi_2^2\end{pmatrix} \qquad \langle 1,\lambda|x\rangle = (\phi_2^2, -\phi_1^2)\\ \langle x|2,\lambda\rangle &= \begin{pmatrix}\psi_2^2(-\lambda)\\ \psi_1^2(-\lambda)\end{pmatrix} \qquad \langle 2,\lambda|x\rangle = (\phi_1^2(-\lambda), -\phi_2^2(-\lambda))\end{aligned} \tag{C39}$$

we have

$$\begin{aligned}\langle x|f\rangle &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{\pi}\left[f(\lambda)\begin{pmatrix}\psi_1^2(\lambda)\\ \psi_2^2(\lambda)\end{pmatrix} + \bar{f}(\lambda)\begin{pmatrix}\psi_2^2(-\lambda)\\ \psi_1^2(-\lambda)\end{pmatrix}\right]\\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{\pi}\left[f(\lambda)\begin{pmatrix}\psi_1^2(\lambda)\\ \psi_2^2(\lambda)\end{pmatrix}\right] + \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{\pi}\left[\bar{f}(\lambda)\begin{pmatrix}\psi_2^2(\lambda)\\ \psi_1^2(\lambda)\end{pmatrix}\right]\end{aligned} \tag{C40}$$

where

$$\bar{f}(-\lambda) = \frac{1}{\bar{a}^2(-\lambda)}\langle 2,-\lambda|f\rangle = \frac{1}{a^2(\lambda)}\int_{-\infty}^{\infty} \mathrm{d}x\,(\phi_1^2 f_1 - \phi_2^2 f_2) \tag{C41}$$

and

$$f(\lambda) = -\frac{1}{a^2(\lambda)}\langle 1,\lambda|f\rangle = \frac{1}{a^2(\lambda)}\int_{-\infty}^{\infty} \mathrm{d}x\,(\phi_1^2 f_2 - \phi_2^2 f_1) \tag{C42}$$

Now we let $f_1(x) = f_2(x) = g(x)$. Thus,

$$\bar{f}(-\lambda) = f(\lambda) = -\frac{1}{a^2(\lambda)}\int_{-\infty}^{\infty} \mathrm{d}x\,(\phi_2^2 - \phi_1^2)g(x) \equiv h(\lambda). \tag{C43}$$

Then the expansion (C40) becomes

$$\begin{aligned}\begin{pmatrix}g(x)\\ g(x)\end{pmatrix} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{\pi}\,h(\lambda)[\langle x|1,\lambda\rangle + \langle x|2,-\lambda\rangle]\\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{\pi}\,h(\lambda)\begin{bmatrix}\psi_1^2(\lambda) + \bar{\psi}_1^2(-\lambda)\\ \psi_2^2(\lambda) + \bar{\psi}_2^2(-\lambda)\end{bmatrix}\\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{\pi}\,h(\lambda)\begin{pmatrix}\psi_1^2 + \psi_2^2\\ \psi_2^2 + \psi_1^2\end{pmatrix}(\lambda).\end{aligned} \tag{C44}$$

Therefore, we have found an expansion for $g(x)$ in terms of the first basis for the perturbed sine-Gordon problem:

$$\begin{aligned}g(x) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{\pi}\,h(\lambda)(\psi_1^2 + \psi_2^2)\\ h(\lambda) &= -\frac{1}{a^2(\lambda)}\int_{-\infty}^{\infty} \mathrm{d}x\,g(x)(\phi_2^2 - \phi_1^2).\end{aligned} \tag{C45}$$

In Kaup's analysis, he started by defining the states (C1)-(C4) in terms of the Jost functions ψ and $\bar{\psi}$. He could just as well have used ϕ and $\bar{\phi}$ to develop his expansions and orthogonality relations. The results of such an analysis would lead to an expansion of $g(x)$ in terms of the second basis, replacing (C45).

As for completeness, Kaup had established the identity operator

$$I = \int_{\bar{C}} \frac{\mathrm{d}\lambda}{\pi} |2,\lambda\rangle \frac{1}{\bar{a}^2(\lambda)} \langle 2,\lambda| - \int_{C} \frac{\mathrm{d}\lambda}{\pi} |1,\lambda\rangle \frac{1}{a^2(\lambda)} \langle 1,\lambda| \tag{C46}$$

(for the case of compact support), and used the Marchenko equations from the inverse scattering formalism to establish that (C46) operates on $L_2(-\infty,\infty)$. Thus, he proved that any $f(x)$ in $L_2(-\infty,\infty)$ can be expanded as in (C16). In particular, we have chosen $f(x) = (g(x), g(x))^{\mathrm{T}}$. Thus, the bases for the perturbation expansion for the sine-Gordon equation are also complete with respect to $L_2(-\infty,\infty)$.

In summary, we have shown that the assumption concerning completeness and the derivation of the orthogonality relations can be proved using Kaup's results in [27]. In a parallel analysis the same statements could be obtained for the bases involved in the perturbation methods for the mKdV and NLS equations.

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